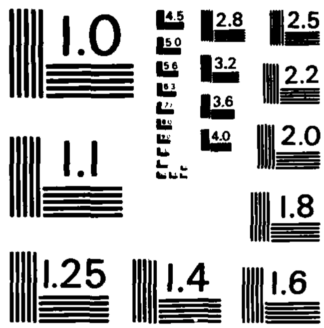
AD-A158 469   GENERAL PERTURBATION SATELLITE THEORY(U) EBELING ASSOCIATES INC SCOTIA NY R H LYDDANE SEP 84 NSWC-TR-84-25 N60921-81-M-D849   1/1

UNCLASSIFIED   F/G 22/3   NL

END
FILMED
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD-F350 044

(2)

AD-A158 469

NSWC TR 84-25

GENERAL PERTURBATION SATELLITE THEORY

BY RUSSELL H. LYDDANE
EBELING ASSOCIATES, INC.

PREPARED FOR
SPACE AND SURFACE SYSTEMS DIVISION
STRATEGIC SYSTEMS DEPARTMENT

SEPTEMBER 1984

DTIC FILE COPY

Approved for public release; distribution unlimited

DTIC
ELECTE
SEP 4 1985
S D
B

NAVAL SURFACE WEAPONS CENTER

Dahlgren, Virginia 22448 • Silver Spring, Maryland 20910

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER NSWC TR 84-25	2. GOVT ACCESSION NO AD-A158469	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) GENERAL PERTURBATION SATELLITE THEORY		5. TYPE OF REPORT & PERIOD COVERED Final
		6. PERFORMING ORG REPORT NUMBER
7. AUTHOR(s) Russell H. Lyddane		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Ebeling Associates, Inc. Scotia, NY 12302		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS N60921-81-M-D849
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Surface Weapons Center (K10) Dahlgren, VA 22448		12. REPORT DATE September 1984
		13. NUMBER OF PAGES 80
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Defense Mapping Agency Washington, DC 20305		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Semianalytic Orbit Theory
Satellite Orbit Theory
General Perturbation Theory

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The theory considers the satellite orbit perturbations due to an arbitrary zonal harmonic gravitational field of the earth and the accelerations due to sun and moon. First-order short-period terms are determined by the von Zeipel technique. The remaining long-periodic equations of motion are integrated numerically with step sizes on the order of 0.25 days.

FOREWORD

The work described in this report was performed by Dr. Russell H. Lyddane of Ebeling Associates, Inc., Scotia, New York. Given are the mathematical derivations and the complete FORTRAN Code.

The Lyddane program is estimated to operate at least two orders of magnitude faster than standard orbit integration techniques. It is eminently suitable for the computation of satellite orbits with moderate accuracy over long periods of time.

Released by:

THOMAS A. CLARE, Head
Strategic Systems Department

DTIC
ELECTE
SEP 4 1985
S D
B

Accession For
NTIS GRA&I
DTIC TAB
Unannounced
Justification
By
Distribution/
Availability Codes
Avail and/or
Dist Special
A-1

DTIC COPY INSPECTED

CONTENTS

INTRODUCTION

The General Perturbation Satellite (GPS) computer program was designed to conduct the integration of the equations of motion of an earth satellite under the influence of (1) an arbitrary zonal-harmonic gravitational field of the earth and (2) the gravitational attractions of the sun and moon. The zonal harmonics (as the program stands) go up to the twentieth, but additional terms can be added by changing a few parameters at the start of the program. There is no provision for tesseral harmonics. Sun and moon coordinates as functions of time are read from a data file that will not be described here, since its construction is outside the program.

The program employs Poincaré-like canonical variables and transforms them by general perturbation theory (von Zeipel) to remove first-order short-period terms (higher-order short-period terms are neglected). The equations in the mean variables thus produced are integrated numerically, with a large step size (set currently at 1/4 day). Since long-period terms receive no special treatment, there is nothing special about the critical angle; and since the variables are Poincaré-like, there are no difficulties at low eccentricities or inclinations.

A listing of the GPS program, in FORTRAN IV, is shown in Appendix A, with line numbers and with concordances for the various subdivisions. This is referenced as appropriate to relate derivations to the final program. The relationships among the subroutines and common variables are shown in Appendix B.

COORDINATES, ELEMENTS, AND CANONICAL VARIABLES

The basic Cartesian coordinate system employed is the nonrotating system with center at the earth's center, Z-axis along the north polar axis, epoch 1950.0. This is the system in which solar and lunar coordinates are also given. The coordinates of the satellite are x, y, z and the velocity components v_x, v_y, v_z.

The (osculating) elliptic elements are the usual set:

a = semi-major axis

e = eccentricity

i = inclination

M = mean anomaly

ω = argument of pericenter

Ω = argument of node

The auxiliary variables follow:

f = true anomaly

E = eccentric anomaly

$\tilde{\omega}$ = $\omega + \Omega$ argument of longitude

α = $E + \tilde{\omega}$ eccentric longitude

u = $f + \tilde{\omega}$ true longitude

The Delaunay canonical variables, used only as convenient intermediate variables, are also as usual:

$$L = \sqrt{\mu a} \qquad \ell = M$$

$$G = L\sqrt{1 - e^2} \qquad g = \omega$$

$$H = G \cos i \qquad h = \Omega$$

where μ is the gravitational constant. Usually ℓ, g, and h will be employed instead of M, ω, and Ω.

The canonical variables used in the program follow:

$$L \qquad \lambda = \ell + \tilde{\omega}$$

$$\xi = \sqrt{2(L-G)}\cos\tilde{\omega} \qquad \eta = -\sqrt{2(L-G)}\sin\tilde{\omega}$$

$$\sigma = \sqrt{2(G-H)}\cos h \qquad \tau = -\sqrt{2(G-H)}\sin h$$

Of course, λ is the mean longitude. Proof that this is indeed a canonical transformation is easily derived (Reference 1).

Several more intermediate quantities are convenient to use with these:

$$\zeta = G/L = \sqrt{1 - e^2} = 1 - (\xi^2 + \eta^2)/2L$$

$$e_c = e\cos\tilde{\omega} = \sqrt{\frac{1+\zeta}{2\,L}}\,\xi$$

$$e_s = e\sin\tilde{\omega} = \sqrt{\frac{1+\zeta}{2\,L}}\,(-\eta)$$

$$\rho = H/G = \cos i = 1 - (\sigma^2 + \tau^2)/2\,\zeta\,L$$

$$I_c = \sin i\cos h = \sigma\sqrt{(1+\rho)/2L\zeta}$$

$$I_s = -\sin i\sin h = \tau\sqrt{(1+\rho/2L\,\zeta}$$

Details of the transformations between these sets of coordinates are given in Appendix C.

---

[1] H. Goldstein, "Classical Mechanics"

MEAN ELEMENTS: THE VON ZEIPEL TRANSFORMATION

The perturbing potential in the Hamiltonian is dominated by the contribution of the second zonal harmonic, whose coefficient is the small perturbing parameter (~ 0.001). All other terms in the perturbing potential are of the order of the square of this parameter. We expand the Hamiltonian in powers of the parameter:

$$F = F_0 + F_1 + F_2 \qquad F_0 = \mu^2/2L^2$$

and introduce a determining function S expanded in the same way:

$$S = S_0 + S_1 + S_2 \qquad \text{where } S_0 = L'\lambda + \xi'\eta + \sigma'\tau$$

and the primed quantities are the mean elements.

We will choose S so that the first-order short-period terms are transformed out. Higher-order short-period terms will be neglected, except for taking into account their effect on the initial value of L', as described below.

Since $\quad F(L, \xi, \sigma, \lambda, \eta, \tau) = F^*(L', \xi', \sigma', -, \eta, \tau)$,

$$F_0\left(\frac{\partial S}{\partial \lambda}\right) + F_1\left(\frac{\partial S}{\partial \lambda}, \frac{\partial S}{\partial \eta}, \frac{\partial S}{\partial \tau}, \lambda, \eta, \tau\right) + F_2 =$$

$$F_0^*(L') + F_1^*\left(L', \xi', \sigma', -, \frac{\partial S}{\partial \xi'}, \frac{\partial S}{\partial \sigma'}\right) + F_2^*$$

Expanding this in the usual Taylor series,

$$F_0(L') + \frac{\partial F_0}{\partial L}(L')\left(\frac{\partial S_1}{\partial \lambda} + \frac{\partial S_2}{\partial \lambda}\right) + \frac{1}{2}\frac{\partial^2 F_0}{\partial L^2}(L')\left(\frac{\partial S_1}{\partial \lambda}\right)^2 + F_1(L', \xi', \sigma', \lambda, \eta, \tau)$$

$$+ \frac{\partial F_1}{\partial L}(L', \xi', \sigma', \lambda, \eta, \tau)\frac{\partial S_1}{\partial \lambda} + \frac{\partial F_1}{\partial \xi}\frac{\partial S_1}{\partial \eta} + \frac{\partial F_1}{\partial \sigma}\frac{\partial S_1}{\partial \tau} + F_2 =$$

$$F_0^*(L') + F_1^*(L', \xi', \sigma', -, \eta, \tau) + \frac{\partial F_1^*}{\partial \eta}\frac{\partial S_1}{\partial \xi} + \frac{\partial F_1^*}{\partial \tau}\frac{\partial S_1}{\partial \sigma} + F_2^*.$$

This can be separated into orders:

$$0: \quad F_0^* = F_0(L') = \mu^2/2L'^2$$

$$\frac{\partial F_0}{\partial L}(L') = -\mu^2/L'^3 \qquad \frac{\partial^2 F_0}{\partial L^2} = 3\mu^2/L'^4$$

$$1: \quad -(\mu^2/L'^3)\frac{\partial S_1}{\partial \lambda} + F_1 = F_1^*$$

Define $F_1 = F_{1s} + F_{1p}$; that is, F_{1s} is the secular and F_{1p} the periodic part. Then,

$$F_1^* = F_{1s}$$

$$\frac{\partial S_1}{\partial \lambda} = \frac{L'^3}{\mu^2} F_{1p}$$

The solution is exactly that of Brouwer (Reference 2); we shall return to it.

$$2: \quad -\frac{\mu^2}{L'^3}\frac{\partial S_2}{\partial \lambda} + \frac{3}{2}\frac{\mu^2}{L'^4}\left(\frac{\partial S_1}{\partial \lambda}\right)^2 + \frac{\partial F_{1s}}{\partial L}\frac{\partial S_1}{\partial \lambda} + \frac{\partial F_{1s}}{\partial \xi}\frac{\partial S_1}{\partial \eta} + \frac{\partial F_{1s}}{\partial \sigma}\frac{\partial S_1}{\partial \tau}$$

$$+ \frac{\partial}{\partial L}\left(\frac{\mu^2}{L^3}\frac{\partial S_1}{\partial \lambda}\right)\frac{\partial S_1}{\partial \lambda} + \frac{\mu^2}{L'^3}\frac{\partial^2 S_1}{\partial \xi \partial \lambda}\frac{\partial S_1}{\partial \eta} + \frac{\mu^2}{L'^3}\frac{\partial^2 S_1}{\partial \sigma \partial \lambda}\frac{\partial S_1}{\partial \tau} + F_2 =$$

$$\frac{\partial F_{1s}}{\partial \eta}\frac{\partial S_1}{\partial \xi} + \frac{\partial F_{1s}}{\partial \tau}\frac{\partial S_1}{\partial \sigma} + F_2^*$$

$F_2{}^*$ is obtained by averaging this equation over λ to remove short-period terms. It consists of the secular part of the summed contributions of the zonal harmonics, omitting the second, plus the solar and lunar contributions, plus the second-order contribution from the second zonal harmonic, which is the same as given in Reference 2, translated into the canonical variables. F_2 is the instantaneous contribution of the perturbing potential, except for the second harmonic.

---

[2] Dirk Brouwer, A. J. 64, 378 (1959)

Since $$L = \frac{\partial S}{\partial \lambda} = L' + \frac{\partial S_1}{\partial \lambda}(L', \xi', \sigma', \lambda, \eta, \tau) + \frac{\partial S_2}{\partial \lambda}$$

$$= L' + \frac{\partial S_1}{\partial \lambda}(L, \xi, \sigma, \lambda, \eta, \tau) - \frac{\partial^2 S_1}{\partial L \partial \lambda}\frac{\partial S_1}{\partial \lambda} - \frac{\partial^2 S_1}{\partial \xi \partial \lambda}\frac{\partial S_1}{\partial \eta} - \frac{\partial^2 S_1}{\partial \sigma \partial \lambda}\frac{\partial S_1}{\partial \tau} + \frac{\partial S_2}{\partial \lambda}.$$

we can solve the second-order part of the Hamiltonian above:

$$\frac{\partial S_2}{\partial \lambda} - \frac{\partial^2 S_1}{\partial L \partial \lambda}\frac{\partial S_1}{\partial \lambda} - \frac{\partial^2 S_1}{\partial \xi \partial \lambda}\frac{\partial S_1}{\partial \eta} - \frac{\partial^2 S_1}{\partial \sigma \partial \lambda}\frac{\partial S_1}{\partial \tau} = - \frac{3}{2L}\left(\frac{\partial S_1}{\partial \lambda}\right)^2$$

$$+ \frac{L^3}{\mu^2}\left(F_2 - F_2^* + \frac{\partial F_{1s}}{\partial L}\frac{\partial S_1}{\partial \lambda} + \frac{\partial F_{1s}}{\partial \xi}\frac{\partial S_1}{\partial \eta} + \frac{\partial F_{1s}}{\partial \sigma}\frac{\partial S_1}{\partial \tau}\right.$$

$$\left. - \frac{\partial F_{1s}}{\partial \eta}\frac{\partial S_1}{\partial \xi} - \frac{\partial F_{1s}}{\partial \tau}\frac{\partial S_1}{\partial \sigma}\right)$$

Substitute in the equation for L:

$$L = L' + \frac{\partial S_1}{\partial \lambda}\left(1 - \frac{3}{2L}\frac{\partial S_1}{\partial \lambda} + \frac{L^3}{\mu^2}\frac{\partial F_{1s}}{\partial L}\right) + \frac{L^3}{\mu^2}\left(F_2 - F_2^* + \frac{\partial F_{1s}}{\partial \xi}\frac{\partial S_1}{\partial \eta}\right.$$

$$\left. + \frac{\partial F_{1s}}{\partial \sigma}\frac{\partial S_1}{\partial \tau} - \frac{\partial F_{1s}}{\partial \eta}\frac{\partial S_1}{\partial \xi} - \frac{\partial F_{1s}}{\partial \tau}\frac{\partial S_1}{\partial \sigma}\right)$$

Now F_{1s} can be readily written [from Brouwer (Reference 2, Equation 13)] in our variables:

$$F_{1s} = (\mu^4 R_e^2 J_2/4 L^6 \zeta^2)(-1 + 3 \rho^2)$$

From Reference 2, Equation 15:

$$S_1 = (6 \mu^2 R_e^2 J_2/(2L \zeta)^3)\left\{\frac{1}{3}(-1 + \rho^2)U_\ell\right.$$

$$\left. + ((1 + \rho)/4 L \zeta)((\sigma^2 - \tau^2) S_{up} + 2\sigma\tau C_{up})\right\}$$

where

$$U_\ell = f - \ell + e \sin f$$

$$S_{up} = \sin 2u + e \sin (u + \tilde{\omega}) + (e/3) \sin (3u - \tilde{\omega})$$

$$C_{up} = \cos 2u + e \cos (u + \tilde{\omega}) + (e/3) \cos (3u - \tilde{\omega})$$

Appendix D contains the tedious details of the calculation of the partial derivatives of S_1 to determine the van Zeipel transformation.

$$L' = L - \frac{\partial S_1}{\partial \lambda} \qquad \lambda' = \lambda + \frac{\partial S_1}{\partial L}$$

$$\xi' = \xi - \frac{\partial S_1}{\partial \eta} \qquad \eta' = \eta + \frac{\partial S_1}{\partial \xi}$$

$$\sigma' = \sigma - \frac{\partial S_1}{\partial \tau} \qquad \tau' = \tau + \frac{\partial S_1}{\partial \sigma}$$

When and only when the first entry is made to the system, it is necessary to compute L' (the initial value) to the second order, in order that $n_0 = \mu^2/L'^3$ does not contain a second-order error (Reference 3). For this case, evaluation of the derivatives of F_{1s} in the equation above leads to

$$\begin{aligned} L \quad = \quad & L' + \frac{\partial S_1}{\partial \lambda}\left(1 - \frac{3}{2L}\frac{\partial S_1}{\partial \lambda} + \gamma'_2(\zeta(1 - 3\rho^2) + 1 + 2\rho - 5\rho^2)\right) \\ & + \frac{L^3}{\mu^2}(F_2 - F_2^*) + \gamma'_2 (1 + 2\rho - 5\rho^2)\left(\eta\frac{\partial S_1}{\partial \xi} - \xi\frac{\partial S_1}{\partial \eta}\right) \\ & + \gamma'_2 \cdot 2\rho\left(\tau\frac{\partial S_1}{\partial \sigma} - \sigma\frac{\partial S_1}{\partial \tau}\right) \end{aligned}$$

Note that this equation is required only for the initial transformation from osculating to mean variables and that in it the derivatives of S are to be evaluated at the point $(L, \xi, \sigma, \lambda, \eta, \tau)$. Any other of the transformations from mean to osculating variables and back again involves only the first derivatives of S_1, and these may be evaluated indifferently in terms of mean or osculating variables, since the difference is second-order short-period and hence to be neglected.

---

[3] R. H. Lyddane and C. J. Cohen, A. J. 67, 176 (1962)

Details of the calculation of the second-order contribution of L' are found in Appendix E.

THE HAMILTONIAN IN TERMS OF MEAN VARIABLES

From this point on, all quantities will be in terms of mean variables and the primes on the variables will be omitted for simplicity of writing. The next task is to average the Hamiltonian over the short period and to put the results into a form suitable for calculation.

ZONAL HARMONICS

We set aside for later consideration the secular second-order contribution of the second zonal harmonic and write F_1^* plus the zonal-harmonic portion of F_2^* as

$$\Delta F_Z = -(\mu/a) \sum_{n=2}^{N} J_n (R_e/a)^n \, \overline{(a/r)^{n+1} \, P_n(\cos\Theta)}$$

This comes from the standard expansion of an arbitrary axially-symmetric gravitational field into spherical harmonics, which is valid as long as $r > R_e$. The bar indicates the average over λ (removal of short-period terms), and Θ is the colatitude of the satellite, that is, the angle between the unit vector to the satellite $\hat{r}$ and the unit vector along the z-axis $\hat{r}_z$:

$$\cos\Theta = \hat{r} \cdot \hat{r}_z$$

The sum is terminated with the highest harmonic to be considered.

The addition theorem for the spherical harmonics,

$$P_n(\cos\Theta) = \sum_{m=0}^{n} (2 - \delta_{m0}) \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta_1) \, P_n^m(\cos\theta_2) \cos m(\varphi_1 - \varphi_2),$$

where $\cos\Theta = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_1 - \varphi_2)$, may be conveniently applied by inserting the representations of $\hat{r}$ and $\hat{r}_z$ in the $\hat{L}_1 \, \hat{L}_2 \, \hat{h}$ frame, namely,

$$\hat{r}: \quad \theta_1 = \pi/2 \qquad\qquad \varphi_1 = u$$

```
C
      SUBROUTINE ENTER(KEY)
C
C     KEY=1:  TRANSFORM FROM COORDINATES TO OSCULATING ELEMENTS
C     KEY=2:  TRANSFORM FROM ELLIPTIC ELEMENTS TO OSCULATING ELEMENTS
C
      DOUBLE PRECISION X,Y,Z,VX,VY,VZ,XI,ETA,SIGMA,TAU,L,LAMBDA,ZETA,
     + L2Z,RHO,C1,C2,ECLON,U,COSU,SINU,ECOSF,ESINF,GAM2P,R,Z9,Z8,IC,IS,
     + HM,EC,ES,A,ECC,INC,ELL,G,H,DRAD,MU
      COMMON /COREL/X,Y,Z,VX,VY,VZ,A,ECC,INC,ELL,G,H/INTEL/ETA,XI,
     + TAU,SIGMA,LAMBDA,L,ZETA,L2Z,RHO,C1,C2,ECLON,U,COSU,SINU,ECOSF,
     + ESINF,EC,ES,GAM2P/COEF/DRAD,MU
C
      IF(KEY.EQ.1) GOTO 100
C
      L=DSQRT(MU*A)
      Z8=(G+H)/DRAD
      LAMBDA=ELL/DRAD+Z8
      ZETA=DSQRT(1-ECC*ECC)
      Z9=DSQRT(2*L*(1-ZETA))
      XI=Z9*DCOS(Z8)
      ETA=-Z9*DSIN(Z8)
      Z9=2*DSIN(INC/DRAD/2)*DSQRT(L*ZETA)
      SIGMA=Z9*DCOS(H/DRAD)
      TAU=-Z9*DSIN(H/DRAD)
      CALL INTERM
      GOTO 200
C
  100 IS=-(Y*VZ-Z*VY)
      IC=-(Z*VX-X*VZ)
      RHO=X*VY-Y*VX
      HM=DSQRT(IS**2+IC**2+RHO**2)
      IS=IS/HM
      IC=IC/HM
      RHO=RHO/HM
      R=DSQRT(X**2+Y**2+Z**2)
      Z9=(X*IS+Y*IC)/(1+RHO)-Z
      U=DATAN2(Y-IC*Z9,X-IS*Z9)
      COSU=DCOS(U)
      SINU=DSIN(U)
      ECOSF=HM*HM/(MU*R)-1
      ESINF=(X*VX+Y*VY+Z*VZ)*HM/(MU*R)
      ZETA=DSQRT(1-ECOSF**2-ESINF**2)
      EC=ECOSF*COSU+ESINF*SINU
      ES=ECOSF*SINU-ESINF*COSU
      Z9=ESINF/(1+ZETA)
      ECLON=DATAN2(SINU+ES-EC*Z9,COSU+EC+ES*Z9)
      LAMBDA=ECLON-EC*DSIN(ECLON)+ES*DCOS(ECLON)
      L=HM/ZETA
      C1=DSQRT((1+ZETA)/(2*L))
      XI=EC/C1
      ETA=-ES/C1
      L2Z=2*L*ZETA
```

NEQ	I //	017740	0005S	0027I	0052	0063	0071	0098	0101	
NTERM	I /COEF/	000030	0004S	0005S	0027I					
OSTCOR	R EXTERNAL	000000	0084	0110						
RE	D /COEF/	000014	0001S	0005S	0025I	0035				
READ	R EXTERNAL	000000	0048	0061	0096					
S	D	001002	0001S	0064M	0065	0067	0102M	0104		
SQRT	R EXTERNAL	000000	0076							
STEP	D	000002	0001S	0027I	0044					
TIMO	D	000006	0001S	0027I						
VAR	D //	000000	0001S	0005S	0051M	0099M				
VEL	D //	000630	0001S	0005S	0053	0064	0099	0102		
VX	D /COREL/	000014	0001S	0005S	0029I	0088	0115			
VY	D /COREL/	000020	0001S	0005S	0029I	0088	0115			
VZ	D /COREL/	000024	0001S	0005S	0029I	0088	0115			
X	D /COREL/	000000	0001S	0005S	0029I	0088	0115			
Y	D /COREL/	000004	0001S	0005S	0029I	0088	0115			
Z	D /COREL/	000010	0001S	0005S	0029I	0088	0115			
Z8	R	001006	0057M	0077						
Z9	R	001010	0037M	0057	0059M	0065M	0076M	0077		
_10		000045	0038	0039	0040	0041D				
_1010		000454	0079D	0088	0115					
_1012		000464	0080D							
_1020		000525	0089D							
_110		000201	0052	0053D						
_120		000224	0057D	0077	0078					
_130		000334	0058	0063	0066	0068D				
_150		000374	0069	0071	0072	0074D				
_20		000123	0043	0045D						
_200		000530	0094D	0108	0116					
_210		000556	0098	0099D						
_220		000670	0103	0105D						
_250		000676	0101	0106D						
_30		000160	0050	0051D						

```
(0107)        KS=KS+1
(0108)        IF(MOD(KS,48).NE.3) GOTO 200
(0109)        CALL ELTRAN(KR,2)
(0110)        CALL OSTCOR
(0111)  C     WRITE(1,1012)KS
(0112)  C     WRITE(1,1010)A,ECC,INC,ELL,G,H
(0113)  C     WRITE(1,1010)(O(I),I=1,6)
(0114)  C     WRITE(1,1010)(VAR(I,KR),I=1,6)
(0115)        WRITE(1,1010)X,Y,Z,VX,VY,VZ
(0116)        IF(KS.LT.1445) GOTO 200
(0117)        CALL EXIT
(0118)        END
```

```
A       D /COREL/   000030   0001S  0005S  0029I
ADAM    D /COEF/    000153   0001S  0005S  0007I  0044M  0045
ADSUM   D           000755   0001S  0042M  0045M  0099
DEL     D //        001460   0001S  0005S  0041M  0053M  0064   0067M  0074M
                             0099   0102   0104M
DERIV   R EXTERNAL  000000   0049   0100
DRAD    D /COEF/    000000   0001S  0005S  0025I
ECC     D /COREL/   000034   0001S  0005S  0029I
ELL     D /COREL/   000044   0001S  0005S  0029I
ELTRAN  R EXTERNAL  000000   0047   0083   0109
ENTER   R EXTERNAL  000000   0046
EXIT    R EXTERNAL  000000   0117
EXTRAP  R EXTERNAL  000000   0062   0097
G       D /COREL/   000050   0001S  0005S  0029I
GAM2N   D /COEF/    000010   0001S  0005S  0035M
H       D /COREL/   000054   0001S  0005S  0029I
HAR     D /COEF/    000033   0001S  0005S  0016I  0035
I       I           000765   0038M  0041   0043M  0044   0045   0050M  0051
INC     D /COREL/   000040   0001S  0005S  0029I
INIT    I           000012   0029I  0046A
J       I           000770   0039M  0041
K       I           000771   0040M  0041   0069M  0070
K9      I           000772   0082M  0083A
KC      I           000773   0066M  0067   0072M  0073   0103M  0104
KC1     I           000774   0073M  0074
KCYC    I           000775   0036M  0061   0075M  0078
KMAX    I //        017741   0005S  0027I  0043   0058   0069   0070   0072
                             0073   0093   0103
KR      I           000776   0058M  0060   0061A  0062A  0064   0066   0067
                             0070M  0074   0095M  0096A  0097A  0099   0100A
                             0102   0104   0109A
KR1     I           000777   0060M  0062A  0094M  0097A
KS      I           001000   0093M  0094   0095   0107M  0108   0116
MM      D /COEF/    000020   0001S  0005S  0025I
MOD     I EXTERNAL  000000   0094   0095   0108
MS      D /COEF/    000024   0001S  0005S  0025I
MU      D /COEF/    000004   0001S  0005S  0025I  0035
N       I           001001   0052M  0053   0063M  0064   0067   0071M  0074
                             0098M  0099   0101M  0102   0104
```

NEQ	I	//	017740	0005S	0027I	0052	0063	0071	0098	0101
NTERM	I	/COEF/	000030	0004S	0005S	0027I				
OSTCOR	R	EXTERNAL	000000	0084	0110					
RE	D	/COEF/	000014	0001S	0005S	0025I	0035			
READ	R	EXTERNAL	000000	0048	0061	0096				
S	D		001002	0001S	0064M	0065	0067	0102M	0104	
SQRT	R	EXTERNAL	000000	0076						
STEP	D		000002	0001S	0027I	0044				
TIMO	D		000006	0001S	0027I					
VAR	D	//	000000	0001S	0005S	0051M	0099M			
VEL	D	//	000630	0001S	0005S	0053	0064	0099	0102	
VX	D	/COREL/	000014	0001S	0005S	0029I	0088	0115		
VY	D	/COREL/	000020	0001S	0005S	0029I	0088	0115		
VZ	D	/COREL/	000024	0001S	0005S	0029I	0088	0115		
X	D	/COREL/	000000	0001S	0005S	0029I	0088	0115		
Y	D	/COREL/	000004	0001S	0005S	0029I	0086	0115		
Z	D	/COREL/	000010	0001S	0005S	0029I	0088	0115		
Z8	R		001006	0057M	0077					
Z9	R		001010	0037M	0057	0059M	0065M	0076M	0077	

_10	000045	0038	0039	0040	0041D
_1010	000454	0079D	0088	0115	
_1012	000464	0080D			
_1020	000525	0089D			
_110	000201	0052	0053D		
_120	000224	0057D	0077	0078	
_130	000334	0058	0063	0066	0068D
_150	000374	0069	0071	0072	0074D
_20	000123	0043	0045D		
_200	000530	0094D	0108	0116	
_210	000556	0098	0099D		
_220	000670	0103	0105D		
_250	000676	0101	0106D		
_30	000160	0050	0051D		

```
(0107)          KS=KS+1
(0108)          IF(MOD(KS,48).NE.3) GOTO 200
(0109)          CALL ELTRAN(KR,2)
(0110)          CALL OSTCOR
(0111)    C     WRITE(1,1012)KS
(0112)    C     WRITE(1,1010)A,ECC,INC,ELL,G,H
(0113)    C     WRITE(1,1010)(O(I),I=1,6)
(0114)    C     WRITE(1,1010)(VAR(I,KR),I=1,6)
(0115)          WRITE(1,1010)X,Y,Z,VX,VY,VZ
(0116)          IF(KS.LT.1445) GOTO 200
(0117)          CALL EXIT
(0118)          END
```

```
A       D /COREL/   000030  0001S  0005S  0029I
ADAM    D /COEF/    000153  0001S  0005S  0007I  0044M  0045
ADSUM   D           000755  0001S  0042M  0045M  0099
DEL     D //        001460  0001S  0005S  0041M  0053M  0064   0067M  0074M
                            0099   0102   0104M
DERIV   R EXTERNAL  000000  0049   0100
DRAD    D /COEF/    000000  0001S  0005S  0025I
ECC     D /COREL/   000034  0001S  0005S  0029I
ELL     D /COREL/   000044  0001S  0005S  0029I
ELTRAN  R EXTERNAL  000000  0047   0083   0109
ENTER   R EXTERNAL  000000  0046
EXIT    R EXTERNAL  000000  0117
EXTRAP  R EXTERNAL  000000  0062   0097
G       D /COREL/   000050  0001S  0005S  0029I
GAM2N   D /COEF/    000010  0001S  0005S  0035M
H       D /COREL/   000054  0001S  0005S  0029I
HAR     D /COEF/    000033  0001S  0005S  0016I  0035
I       I           000765  0038M  0041   0043M  0044   0045   0050M  0051
INC     D /COREL/   000040  0001S  0005S  0029I
INIT    I           000012  0029I  0046A
J       I           000770  0039M  0041
K       I           000771  0040M  0041   0069M  0070
K9      I           000772  0082M  0083A
KC      I           000773  0066M  0067   0072M  0073   0103M  0104
KC1     I           000774  0073M  0074
KCYC    I           000775  0036M  0061   0075M  0078
KMAX    I //        017741  0005S  0027I  0043   0058   0069   0070   0072
                            0073   0093   0103
KR      I           000776  0058M  0060   0061A  0062A  0064   0066   0067
                            0070M  0074   0095M  0096A  0097A  0099   0100A
                            0102   0104   0109A
KR1     I           000777  0060M  0062A  0094M  0097A
KS      I           001000  0093M  0094   0095   0107M  0108   0116
MM      D /COEF/    000020  0001S  0005S  0025I
MOD     I EXTERNAL  000000  0094   0095   0108
MS      D /COEF/    000024  0001S  0005S  0025I
MU      D /COEF/    000004  0001S  0005S  0025I  0035
N       I           001001  0052M  0053   0063M  0064   0067   0071M  0074
                            0098M  0099   0101M  0102   0104
```

```
C
C     CYCLE ON STARTING SOLUTION
C
120   Z8=Z9
      DO 130 KR=2,KMAX
      Z9=0
      KR1=KR-1
      IF(KCYC.EQ.0)CALL READ(KR)
      CALL EXTRAP(KR1,KR)
      DO 130 N=1,NEQ
      S=VEL(N,KR)-DEL(1,N,KR)
      Z9=Z9+S*S
      DO 130 KC=1,KR
      DEL(KC,N,KR)=DEL(KC,N,KR)+S
130   CONTINUE
      DO 150 K=2,KMAX
      KR=KMAX+1-K
      DO 150 N=1,NEQ
      DO 150 KC=2,KMAX
      KC1=KMAX+2-KC
150   DEL(KC1,N,KR)=DEL(KC1,N,KR+1)-DEL(KC1+1,N,KR+1)
      KCYC=KCYC+1
      Z9=SQRT(Z9)
      IF (Z9.LT.Z8)GOTO 120
      IF(KCYC.LT.6)GOTO 120
1010  FORMAT(1HX,6D18.10)
1012  FORMAT(3I5)
C     DO 1020 K9=1,KMAX
      K9=2
      CALL ELTRAN(K9,2)
      CALL OSTCOR
C     WRITE (1,1010)A,ECC,INC,ELL,G,H
C     WRITE(1,1010)(O(K),K=1,6)
C     WRITE(1,1010)(VAR(K,K9),K=1,6)
      WRITE(1,1010)X,Y,Z,VX,VY,VZ
1020  CONTINUE
C
C     END OF CYCLE: CONTINUE THE SOLUTION
C
      KS=KMAX+1
200   KR1=MOD(KS-2,17)+1
      KR=MOD(KS-1,17)+1
      CALL READ(KR)
      CALL EXTRAP(KR1,KR)
      DO 210 N=1,NEQ
210   VAR(N,KR)=VAR(N,KR)+ADSUM*(VEL(N,KR)-DEL(1,N,KR))
      CALL DERIV(KR,1)
      DO 250 N=1,NEQ
      S=VEL(N,KR)-DEL(1,N,KR)
      DO 220 KC=1,KMAX
      DEL(KC,N,KR)=DEL(KC,N,KR)+S
220   CONTINUE
250   CONTINUE
```

```
      DOUBLE PRECISION VAR(6,17),VEL(6,17),ADAM(17),DEL(18,6,17),
     + STEP,S,TIM,TIMO,ADSUM,DRAD,MU,RE,MM,MS,GAM2N,X,Y,Z,VX,VY,VZ,
     + A,ECC,INC,ELL,G,H,HAR(20),O(6)
      DIMENSION NTERM(3)
      COMMON //VAR,VEL,DEL,NEQ,KMAX/COEF/DRAD,MU,GAM2N,RE,MM,MS,NTERM,
     + HAR,ADAM/COREL/X,Y,Z,VX,VY,VZ,A,ECC,INC,ELL,G,H,O
      DATA ADAM(1),ADAM(2),ADAM(3),ADAM(4),ADAM(5),ADAM(6),ADAM(7),
     + ADAM(8),ADAM(9),ADAM(10),ADAM(11),ADAM(12),ADAM(13),ADAM(14),
     + ADAM(15),ADAM(16),ADAM(17)/
     + -0.10000000000000D 01,0.50000000000000D 00,0.83333333333333D-01,
     + 0.41666666666667D-01,0.26388888888889D-01,0.18750000000000D-01,
     + 0.14269179894180D-01,0.11367394179894D-01,0.93565365961199D-02,
     + 0.78925540123457D-02,0.67858499846347D-02,0.59240564123377D-02,
     + 0.52366932579503D-02,0.46774984070423D-02,0.42149522390055D-02,
     + 0.38268995532120D-02,0.34973498453500D-02/
      DATA HAR(2),HAR(3),HAR(4),HAR(5),HAR(6),HAR(7),HAR(8),HAR(9),
     + HAR(10),HAR(11),HAR(12),HAR(13),HAR(14),HAR(15),HAR(16),HAR(17),
     + HAR(18),HAR(19),HAR(20)/
     +     0.108263 4 D-02,        -0.2536 D-05,
     +    -0.1664 D-05,            -0.2195 D-06,
     +     0.6355 D-06,            -0.3720 D-06,
     +    -0.3508 D-06,            -0.8733 D-07,
     +    -0.5730 D-07,             0.1686 D-06,
     +    -0.3809 D-06,      0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0/
      DATA MU,RE,MM,MS,DRAD/.398601 178977 8 D06,.637814 5 D04,.122999
     + 7171 D-01,.332945 561925 44 D06,.572957 795130 823 D02/
      DATA NEQ,KMAX,STEP,TIMO,NTERM(1),NTERM(2),NTERM(3)/5,9,.216D5,
     + 0.D0,12,5,2/
      DATA INIT,X,Y,Z,VX,VY,VZ,A,ECC,INC,ELL,G,H/2,0,0,0,0,0,0,
     + 0.748503 712201 72 D04, 0.8255 D-02, 0.634300 470727 88 D02,
     + 0.1033005 D03, 0.19952 D03, 0.1249632 D03/
C
C     INITIALIZE
C
      GAM2N=12*HAR(2)*(MU*RE)**2
      KCYC=0
      Z9=1.E20
      DO 10 I=1,17
      DO 10 J=1,6
      DO 10 K=1,18
   10 DEL(K,J,I)=0
      ADSUM=0
      DO 20 I=1,KMAX
      ADAM(I)=-STEP*ADAM(I)
   20 ADSUM=ADSUM+ADAM(I)
      CALL ENTER(INIT)
      CALL ELTRAN(1,1)
      CALL READ(1)
      CALL DERIV(1,2)
      DO 30 I=1,17
   30 VAR(6,I)=VAR(6,1)
      DO 110 N=1,NEQ
  110 DEL(1,N,1)=VEL(N,1)
```


PREVIOUS PAGE IS BLANK

APPENDIX A

GPS PROGRAM LISTING

where the only zero-order term is μ^2/L^3, a constant. There are, thus, five simultaneous first-order differential equations to be integrated step by step (L being a constant of the motion), after the partial derivatives of F_1 have been evaluated at each step.

The program (in the subroutine DERIV) defines the array

$$DF(I) = \left(\frac{\partial F_1}{\partial \alpha}, \frac{\partial F_1}{\partial \beta}, \frac{\partial F_1}{\partial \gamma}, \frac{\partial F_1}{\partial \zeta}, \frac{\partial F_1}{\partial a}\right)$$

whose elements are defined piecemeal in the previous section since $\partial/\partial\zeta = (-1/\zeta^2)(\partial/\partial x)$. The right-hand sides of the equations of motion are contained in the array

$$DE(I) = \left(\frac{\partial F_1}{\partial \eta}, \frac{\partial F_1}{\partial \xi}, \frac{\partial F_1}{\partial \tau}, \frac{\partial F_1}{\partial \sigma}, \frac{\partial F_1}{\partial L}\right)$$

The connection between these two is the Jacobian matrix

$$\frac{J(\alpha, \beta, \gamma, \zeta, a)}{J(\eta, \xi, \tau, \sigma, L)}$$

The components of the matrix are evaluated in Appendix G. The program equates the matrix to the array C(I, J), where for example $C(3, 4) = \partial\gamma/\partial\sigma$. Then clearly,

$$DE(I) = \sum_J DF(J)\ C(J, I)$$

is the set of time derivatives of the variables, and this is the output of DERIV.

The equations of motion are integrated by a standard Adams numerical integration routine. Although provision is made for an order as high as 17, experimentation indicated that an order of 9 gave the best compromise between roundoff error and truncation error, so, the program is currently set to 9 (KMAX Value). The starting procedure is found in the main program (lines 57-78). At least 6 cycles are performed, and more if the rms change in the derivatives per cycle continues to decline. After the starting values are stabilized, the running procedure applies a predictor-corrector pair once per time step. The extrapolation forward of the backward difference table and the application of the predictor are to be found in the subroutine EXTRAP; the corrector is applied in the main program, lines 94-108.

$\frac{\partial}{\partial \beta} \Delta F_M$	C_m	$-mS_{m-1}$
$\frac{\partial}{\partial \gamma} \Delta F_M$	Q_n^m	Q_n^{m+1}
$\frac{\partial}{\partial x} \Delta F_M$	A_{n+1}^m	$\frac{-n+m}{x} (A_{n+1}^m + A_{n+1}^{m+1})$
$\frac{\partial}{\partial a} \Delta F_M$	$\left(\frac{a}{r_M}\right)^n$	$\frac{n}{a}\left(\frac{a}{r_M}\right)^n$

THE SOLAR ATTRACTION

The contribution of the sun is exactly the same as the moon's except that μ_S and r_S are substituted for μ_M and r_M and x'_S, y'_S, z'_S are substituted for x'_M, y'_M, z'_M.

THE EQUATIONS OF MOTION IN THE MEAN VARIABLES

The Hamiltonian in the mean variables may be written

$$F = \mu^2/2L^2 + F_1$$

$$F_1 = \Delta F_Z + \Delta F_{2s} + \Delta F_M + \Delta F_S$$

Since the variables are canonical, the Hamiltonian equations of motion are

$$\dot{\eta} = -\frac{\partial F_1}{\partial \xi} \qquad \dot{\xi} = \frac{\partial F_1}{\partial \eta}$$

$$\dot{\tau} = -\frac{\partial F_1}{\partial \sigma} \qquad \dot{\sigma} = \frac{\partial F_1}{\partial \tau}$$

$$\dot{\lambda} = \mu^2/L^3 - \frac{\partial F_1}{\partial L} \qquad \dot{L} = 0$$

THE LUNAR ATTRACTION

The moon's contribution to the Hamiltonian can also be expanded into a spherical harmonic series and averaged over the short period:

$$F_M = \left(\frac{\mu_M}{r_M}\right) \sum_{n=2} \left(\frac{a}{r_M}\right)^n \overline{\left(\frac{r}{a}\right)^n P_n \ (\cos \Theta)}$$

Here $\cos \Theta = \hat{r} \cdot \hat{r}_M$, i.e., Θ is the angle between the radius vector of the satellite and the vector from the origin to the moon. The distance to the moon is r_M, and the components of the unit vector $\hat{r}_M$ are x'_M, y'_M, z'_M. The expansion is valid as long as $r < r_M$ and the series is terminated when the terms become negligibly small.

The development of this contribution is exactly like that for the zonal harmonics, with two exceptions: α, β, and γ contain x'_M, y'_M, z'_M in place of x'_Z y'_Z, z'_Z and the coefficients in x, instead of B_n^m (x), are defined by

$$\overline{\left(\frac{r}{a}\right)^n \cos mf} = e^m \ A_{n+2}{}^m \ (x)$$

The upshot is

$$\Delta F_M = \frac{\mu_M}{r_M} \sum_{m=0}^{N} (2 - \delta_{m0}) \ C_m \ (\alpha, \beta) \sum_n \left(\frac{a}{r_M}\right)^n \ D_n^m \ Q_n^m \ (\gamma) \ A_{n+2}^m \ (x)$$

The sum over m being from max (m, 2) by steps of 2 to N.

The recursion relationships for D_n^m, Q_n^m (γ), and C_m (α, β) are exactly the same as for the zonal case, although of course the arguments are different. The recursions and the derivatives of the recursive functions are given in Appendix F. The derivatives of ΔF_M can be written as follows:

In ΔF_M, obtain:

$\frac{\partial}{\partial \alpha} \Delta F_M$ by replacing C_m with mC_{m-1}

$$\frac{\partial}{\partial a} \Delta F_z \qquad\qquad \frac{\mu}{a}\left(\frac{R_e}{a}\right)^n \qquad\qquad -\frac{n+1}{a}\frac{\mu}{a}\left(\frac{R_e}{a}\right)^n$$

THE SECULAR SECOND-ORDER CONTRIBUTION OF THE SECOND HARMONIC

Brouwer's Equation 29 (Reference 2) can readily be put in terms of the variables being used here:

$$\Delta F_{2s} = (\mu^2/L^3)\ \gamma_2'^2\ L(\zeta/24) \Big\{ 5\ \zeta^2\ (1 - 18\ \rho^2/5 + \rho^4)$$

$$+ 4\ \zeta\ (1-6\ \rho^2 + 9\rho^4) - 5\ (1-2\ \rho^2 - 7\ \rho^4) - 2(1-15\ \rho^2)(\alpha^2 - \beta^2) \Big\}$$

It follows that, if $M = (\mu^2/L^3)\ \gamma_2'^2\ L\ \zeta/24$ and $\gamma = \rho$,

$$\frac{\partial}{\partial \alpha} \Delta F_{2s} = M(-\ 4\alpha)(1-15\ \rho^2)$$

$$\frac{\partial}{\partial \beta} \Delta F_{2s} = M(4\ \beta)(1-15\ \rho^2)$$

$$\frac{\partial}{\partial \gamma} \Delta F_{2s} = M(4\ \rho) \Big\{ \zeta^2\ (-9 + 5\rho^2) + 12\ \zeta\ (-1 + 3\ \rho^2)$$

$$+ 5\ (1 + 7\ \rho^2) + 15\ (\alpha^2 - \beta^2) \Big\}$$

$$\frac{\partial}{\partial x} \Delta F_{2s} = M(-\ \zeta^2) \Big\{ 10\ \zeta\ (1-18\ \rho^2/5 + \rho^4) + 4\ (1-6\ \rho^2 + 9\ \rho^4) \Big\}$$

$$+ 7\ \zeta \cdot \Delta F_{2s}$$

$$\frac{\partial}{\partial a} \Delta F_{2s} = -\ (5/a)\ \Delta F_{2s}$$

Then, $\alpha = e_c \ \psi_1 + e_s \ \psi_2$

$$\beta = e_s \ \psi_1 - e_c \ \psi_2$$

and $C_m \ (\alpha, \beta) = \text{Re} \ [(\alpha + i \ \beta)^m] = e^m \ (1 - \gamma^2)^{m/2} \cos m \ (\widetilde{\omega} - \varphi_2)$

Introduce for convenience one more set of polynomials by

$$D_n^m = \frac{(n-m)\ !}{(n+m)\ !} Q_n^m \ (0)$$

and introduce the new variable $x = 1/\zeta$.

Then,

$$\Delta F_z = - \sum_{m=0}^{N} (2 - \delta_{m0}) \ C_m \ (\alpha, \beta) \sum_n J_n \ (\mu/a) \ (R_e/a)^n \ D_n^m \ Q_n^m \ (\gamma) \ B_n^m \ (x)$$

The sum over n starts at 2 or m, whichever is greater, and goes in steps of 2 to N.

This formulation has the advantages that the Hamiltonian depends only on five intermediate variables [α, β, γ, ζ (or x), and a] through three functions (Q_n^m, B_n^m, and C_m) and one set of constants (D_n^m), all of which can be calculated recursively; and the partial derivatives are reasonably simple.

The recursion relationships and the derivatives of the functions are developed in Appendix F. The derivatives of ΔF_z can, with the aid of those results, be written as the following schema:

In ΔF_z, obtain:

$\frac{\partial}{\partial \alpha} \Delta F_z$ by replacing	C_m	with	mC_{m-1}
$\frac{\partial}{\partial \beta} \Delta F_z$	C_m		$-m \ S_{m-1}$
$\frac{\partial}{\partial \gamma} \Delta F_z$	Q_n^m		Q_n^{m+1}
$\frac{\partial}{\partial x} \Delta F_z$	B_n^m		$\frac{n+m}{x} (B_n^m + B_n^{m+1})$

$$\hat{r}_z:\ \sin\theta_2 \cos\varphi_2 = \psi_1 = \hat{L}_1 \cdot \hat{r}_z = x'_z + \frac{I_s}{1+\rho}(\gamma + z'_z)$$

$$\sin\theta_2 \sin\varphi_2 = \psi_2 = \hat{L}_2 \cdot \hat{r}_z = y'_z + \frac{I_c}{1+\rho}(\gamma + z'_z)$$

$$\cos\theta_2 = \gamma = \hat{h} \cdot \hat{r}_z = -I_s x'_z - I_c y'_z + \rho z'_z$$

where the unit vector $\hat{r}_z = \begin{pmatrix} x'_z \\ y'_z \\ z'_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(There is an obvious simplification if the actual values for x'_z, y'_z, z'_z are substituted in the foregoing, but this form is used for parallelism with the lunar and solar contributions to the Hamiltonian.)

$$P_n(\cos\Theta) = \sum_{m=0}^{n} (2 - \delta_{mo}) \frac{(n-m)!}{(n+m)!} P_n^m(0) P_n^m(\gamma) \cos m(u - \varphi_2)$$

If $Q_n^m(\gamma) = d^m/d\gamma^m\ P_n(\gamma)$, then $P_n^m(\gamma) = (1-\gamma^2)^{m/2} Q_n^m(\gamma)$ and

$$\overline{(a/r)^{n+1} P_n(\cos\Theta)} = \sum_m (2 - \delta_{mo}) \frac{(n-m)!}{(n+m)!} Q_n^m(0) Q_n^m(\gamma)\ \overline{(a/r)^{n+1} \cos mf} \cdot$$

$$(1-\gamma^2)^{m/2} \cos m(\tilde{\omega} - \varphi_2)$$

We define B_n^m by $\overline{(a/r)^{n+1} \cos mf} = e^m B_n^m$ (knowing that the Cayley coefficients are proportional to e^m) and consider

$$[e\sqrt{1-\gamma^2} \exp i(\tilde{\omega} - \varphi_2)]^m = [(e_c + i e_s)(\psi_1 - i\psi_2)]^m$$

$$= [e_c \psi_1 + e_s \psi_2 + i(e_s \psi_1 - e_c \psi_2)]^m$$

Define polynomials C_m and S_m by

$$(\alpha + i\beta)^m = C_m(\alpha, \beta) + i S_m(\alpha, \beta)$$

```
(0172)        C2=DSQRT((1+RHO)/L2Z)
(0173)        SIGMA=IC/C2
(0174)        TAU=IS/C2
(0175)  C
(0176)  200   CALL XFR(0,3)
(0177)        RETURN
(0178)        END
```

```
A       D /COREL/   000030   0125S  0128S  0134
C1      D /INTEL/   000044   0125S  0128S  0168M  0169   0170
C2      D /INTEL/   000050   0125S  0128S  0172M  0173   0174
COSU    D /INTEL/   000064   0125S  0128S  0157M  0162   0163   0165
DATAN2  D EXTERNAL  000000   0156   0165
DCOS    D EXTERNAL  000000   0139   0142   0157   0166
DRAD    D /COEF/    000000   0125S  0128S  0135   0136   0141   0142   0143
DSIN    D EXTERNAL  000000   0140   0141   0143   0158   0166
DSQRT   D EXTERNAL  000000   0134   0137   0138   0141   0150   0154   0161
                             0168   0172
EC      D /INTEL/   000104   0125S  0128S  0162M  0165   0166   0169
ECC     D /COREL/   000034   0125S  0128S  0137
ECLON   D /INTEL/   000054   0125S  0128S  0165M  0166A
ECOSF   D /INTEL/   000074   0125S  0128S  0159M  0161   0162   0163
ELL     D /COREL/   000044   0125S  0128S  0136
ES      D /INTEL/   000110   0125S  0128S  0163M  0165   0166   0170
ESINF   D /INTEL/   000100   0125S  0128S  0160M  0161   0162   0163   0164
ETA     D /INTEL/   000000   0125S  0128S  0140M  0170M
G       D /COREL/   000050   0125S  0128S  0135
H       D /COREL/   000054   0125S  0128S  0135   0142   0143
HM      D           001007   0125S  0150M  0151   0152   0153   0159   0160
                             0167
IC      D           001013   0125S  0148M  0150   0152M  0155   0156   0173
INC     D /COREL/   000040   0125S  0128S  0141
INTERM  I EXTERNAL  000000   0144
IS      D           001017   0125S  0147M  0150   0151M  0155   0156   0174
KEY     I ARGUMENT  000003   0120S  0132
L       D /INTEL/   000024   0125S  0128S  0134M  0138   0141   0167M  0168
                             0171
L2Z     D /INTEL/   000034   0125S  0128S  0171M  0172
LAMBDA  D /INTEL/   000020   0125S  0128S  0136M  0166M
MU      D /COEF/    000004   0125S  0128S  0134   0159   0160
R       D           001023   0125S  0154M  0159   0160
RHO     D /INTEL/   000040   0125S  0128S  0149M  0150   0153M  0155   0172
SIGMA   D /INTEL/   000014   0125S  0128S  0142M  0173M
SINU    D /INTEL/   000070   0125S  0128S  0158M  0162   0163   0165
TAU     D /INTEL/   000010   0125S  0128S  0143M  0174M
U       D /INTEL/   000060   0125S  0128S  0156M  0157A  0158A
VX      D /COREL/   000014   0125S  0128S  0148   0149   0160
VY      D /COREL/   000020   0125S  0128S  0147   0149   0160
VZ      D /COREL/   000024   0125S  0128S  0147   0148   0160
X       D /COREL/   000000   0125S  0128S  0148   0149   0154   0155   0156
                             0160
```

```
XFR     R EXTERNAL 000000   0176
XI      D /INTEL/  000004   0125S  0128S  0139M  0169M
Y       D /COREL/  000004   0125S  0128S  0147   0149   0154   0155   0156
                            0160
Z       D /COREL/  000010   0125S  0128S  0147   0148   0154   0155   0160
Z8      D          001027   0125S  0135M  0136   0139A  0140A
Z9      D          001033   0125S  0138M  0139   0140   0141M  0142   0143
                            0155M  0156   0164M  0165
ZETA    D /INTEL/  000030   0125S  0128S  0137M  0138   0141   0161M  0164
                            0167   0168   0171

_100               000176   0132   0147D
_200               000766   0145   0176D
```

```
(0179)  C
(0180)        SUBROUTINE OSTCOR
(0181)  C
(0182)  C     TRANSFORM FROM OSCULATING ELEMENTS TO COORDINATES/ELLIPTIC ELEMENTS
(0183)  C
(0184)        DOUBLE PRECISION X,Y,Z,VX,VY,VZ,XI,ETA,SIGMA,TAU,L,LAMBDA,ZETA,
(0185)       + L2Z,RHO,C1,C2,ECLON,U,COSU,SINU,ECOSF,ESINF,R,Z9,EC,ES,
(0186)       + A,ECC,INC,ELL,G,H,DRAD,MU
(0187)        COMMON /COREL/X,Y,Z,VX,VY,VZ,A,ECC,INC,ELL,G,H/INTEL/ETA,XI,
(0188)       + TAU,SIGMA,LAMBDA,L,ZETA,L2Z,RHO,C1,C2,ECLON,U,COSU,SINU,ECOSF,
(0189)       + ESINF,EC,ES/COEF/DRAD,MU
(0190)  C
(0191)        CALL XFR(0,1)
(0192)        CALL INTERM
(0193)  C
(0194)        A=L*L/MU
(0195)        ECC=DSQRT(1-ZETA*ZETA)
(0196)        Z9=DSQRT(1-RHO*RHO)
(0197)        INC=DRAD*DATAN2(Z9,RHO)
(0198)        IF(SIGMA.NE.0.OR.TAU.NE.0) H=DRAD*DATAN2(-TAU,SIGMA)
(0199)        G=DRAD*DATAN2(-ETA,XI)-H
(0200)        ELL=DRAD*LAMBDA-G-H
(0201)        ELL=DMOD(ELL,3.6D2)
(0202)        IF(ELL.LT.0.D0)ELL=ELL+3.6D2
(0203)        IF(ELL.GT.1.8D2)ELL=ELL-3.6D2
(0204)  C
(0205)        R=(1-EC*DCOS(ECLON)-ES*DSIN(ECLON))*L*L/MU
(0206)        X=R*(COSU*(1-TAU*TAU/L2Z)-SINU*SIGMA*TAU/L2Z)
(0207)        Y=R*(-COSU*SIGMA*TAU/L2Z+SINU*(1-SIGMA*SIGMA/L2Z))
(0208)        Z9=DSQRT((1+RHO)/L2Z)
(0209)        Z=R*Z9*(SINU*SIGMA+COSU*TAU)
(0210)        VX=((ECLON-LAMBDA)*X+ZETA*(-C2*SIGMA*Z-RHO*Y))*L/(R*R)
(0211)        VY=((ECLON-LAMBDA)*Y+ZETA*(RHO*X+C2*TAU*Z))*L/(R*R)
(0212)        VZ=((ECLON-LAMBDA)*Z+ZETA*(-C2*TAU*Y+C2*SIGMA*X))*L/(R*R)
(0213)        RETURN
(0214)        END
```

A	D	/COREL/	000030	0184S	0187S	0194M			
C2	D	/INTEL/	000050	0184S	0187S	0210	0211	0212	
COSU	D	/INTEL/	000064	0184S	0187S	0206	0207	0209	
DATAN2	D	EXTERNAL	000000	0197	0198	0199			
DCOS	D	EXTERNAL	000000	0205					
DMOD	D	EXTERNAL	000000	0201					
DRAD	D	/COEF/	000000	0184S	0187S	0197	0198	0199	0200
DSIN	D	EXTERNAL	000000	0205					
DSQRT	D	EXTERNAL	000000	0195	0196	0208			
EC	D	/INTEL/	000104	0184S	0187S	0205			
ECC	D	/COREL/	000034	0184S	0187S	0195M			
ECLON	D	/INTEL/	000054	0184S	0187S	0205A	0210	0211	0212
ELL	D	/COREL/	000044	0184S	0187S	0200M	0201M	0202M	0203M
ES	D	/INTEL/	000110	0184S	0187S	0205			
ETA	D	/INTEL/	000000	0184S	0187S	0199			
G	D	/COREL/	000050	0184S	0187S	0199M	0200		

```
H       D /COREL/   000054   0184S   0187S   0198M   0199    0200
INC     D /COREL/   000040   0184S   0187S   0197M
INTERM  I EXTERNAL  000000   0192
L       D /INTEL/   000024   0184S   0187S   0194    0205    0210    0211    0212
L2Z     D /INTEL/   000034   0184S   0187S   0206    0207    0208
LAMBDA  D /INTEL/   000020   0184S   0187S   0200    0210    0211    0212
MU      D /COEF/    000004   0184S   0187S   0194    0205
R       D           000535   0184S   0205M   0206    0207    0209    0210    0211
                             0212
RHO     D /INTEL/   000040   0184S   0187S   0196    0197A   0208    0210    0211
SIGMA   D /INTEL/   000014   0184S   0187S   0198A   0206    0207    0209    0210
                             0212
SINU    D /INTEL/   000070   0184S   0187S   0206    0207    0209
TAU     D /INTEL/   000010   0184S   0187S   0198    0206    0207    0209    0211
                             0212
VX      D /COREL/   000014   0184S   0187S   0210M
VY      D /COREL/   000020   0184S   0187S   0211M
VZ      D /COREL/   000024   0184S   0187S   0212M
X       D /COREL/   000000   0184S   0187S   0206M   0210    0211    0212
XFR     R EXTERNAL  000000   0191
XI      D /INTEL/   000004   0184S   0187S   0199A
Y       D /COREL/   000004   0184S   0187S   0207M   0210    0211    0212
Z       D /COREL/   000010   0184S   0187S   0209M   0210    0211    0212
Z9      D           000541   0184S   0196M   0197A   0208M   0209
ZETA    D /INTEL/   000030   0184S   0187S   0195    0210    0211    0212
```

```
C
      SUBROUTINE ELTRAN(KROW,KEY)
C
C     KEY=1:  TRANSFORM FROM OSCULATING TO MEAN ELEMENTS
C     KEY=2:  TRANSFORM FROM MEAN TO OSCULATING ELEMENTS
C
      DOUBLE PRECISION XI,ETA,SIGMA,TAU,L,LAMBDA,ZETA,
     + L2Z,RHO,C1,C2,ECLON,U,COSU,SINU,ECOSF,ESINF,GAM2P,C(6),D(6),
     + E(6),O(6),ZETA2,RHO3,Z9,Z8,Z7,Z6,Z5,Z4,Z,C2U,S2U,SU3,CU3,UL,
     + SUP,CUP,SUPS2,SGH,EC,ES,VAR(6,17),DRAD,MU
      COMMON /COREL/C,E,O//VAR/INTEL/ETA,XI,TAU,SIGMA,LAMBDA,L,
     + ZETA,L2Z,RHO,C1,C2,ECLON,U,COSU,SINU,ECOSF,ESINF,EC,ES,GAM2P
     + /COEF/DRAD,MU
C
      CALL XFR(KROW,KEY)
   20 CALL INTERM
      ZETA2=ZETA*ZETA
      Z=1+ECOSF
      RHO3=(-1+3*RHO*RHO)/3
      Z9=SIGMA*SIGMA-TAU*TAU
      Z8=2*SIGMA*TAU
      Z7=DSIN(2*U)
      Z6=DCOS(2*U)
      Z5=SINU+DSIN(3*U)/3
      Z4=COSU+DCOS(3*U)/3
      C2U=Z6*Z9-Z7*Z8
      S2U=Z7*Z9+Z6*Z8
      SU3=Z5*Z9+Z4*Z8
      CU3=(2*COSU-Z4)*Z9+(-2*SINU+Z5)*Z8
      UL=U-LAMBDA+ESINF
      SUP=Z7+EC*Z5+ES*(2*COSU-Z4)
      CUP=Z6+EC*Z4+ES*(Z5-2*SINU)
      SUPS2=SUP*Z9+CUP*Z8
      SGH=(1+RHO*(2-5*RHO))*UL-(3+5*RHO)*SUPS2/(2*L2Z)
      Z9=Z*(1+Z)*(RHO3+(1+RHO)/L2Z*C2U)
      D(5)=-GAM2P*(SGH-(ESINF*Z9+ZETA2*(RHO3*ESINF+(1+RHO)/(2*L2Z)*
     + (SUPS2-S2U)))/(1+ZETA))
      Z8=(Z/ZETA)**3
      D(6)=GAM2P*(RHO3*(Z8-1)*L2Z+(1+RHO)*Z8*C2U)/2
      Z9=(RHO3+Z*(1+RHO)*C2U/L2Z)/(2*ZETA2)
      Z8=(1+Z)*Z9+RHO3*(Z/ZETA)**2/2
      Z7=ESINF*(1+2*ZETA)*Z8+(1+RHO)*(SUPS2-S2U)/(8*L)
      Z7=SGH+RHO3+Z7/(1+ZETA)
      Z6=(1+ZETA+ZETA2)*Z9
      Z5=(1+RHO)*C1/4
      Z9=2*L*C1*Z8
      D(1)=-GAM2P*(-XI*Z7+Z9*SINU-Z6*ETA+Z5*SU3)
      D(2)=GAM2P*(-ETA*Z7+Z9*COSU+Z6*XI-Z5*CU3)
      Z9=2*RHO*UL+SUPS2/(2*L2Z)
      D(3)=-GAM2P*(-SIGMA*Z9+(1+RHO)/2*(SIGMA*SUP+TAU*CUP))
      D(4)=GAM2P*(-TAU*Z9+(1+RHO)/2*(-TAU*SUP+SIGMA*CUP))
C
      IF(KEY.EQ.1)GOTO 100
```

```
(0268) C
(0269)        DO 50 I=1,6
(0270) 50     O(I)=VAR(I,KROW)+D(I)
(0271)        RETURN
(0272) C
(0273) 100    Z8=1+RHO*(2-5*RHO)
(0274)        D(6)=D(6)*(1-3/(2*L)*D(6)+GAM2P*(-3*ZETA*RHO3+Z8))+
(0275)       + GAM2P*(-Z8*(ETA*D(1)+XI*D(2))-2*RHO*(TAU*D(3)+SIGMA*D(4)))
(0276)        DO 150 I=1,6
(0277) 150    VAR(I,KROW)=O(I)-D(I)
(0278)        RETURN
(0279)        END
```

```
 C1      D /INTEL/   000044    0221S   0225S   0259    0260
 C2U     D           001376    0221S   0240M   0249    0253    0254
 COSU    D /INTEL/   000064    0221S   0225S   0239    0243    0245    0262
 CU3     D           001402    0221S   0243M   0262
 CUP     D           001406    0221S   0246M   0247    0264    0265
 D       D           000006    0221S   0250M   0253M   0261M   0262M   0264M   0265M
                               0270    0274M   0277
 DCOS    D EXTERNAL  000000    0237    0239
 DSIN    D EXTERNAL  000000    0236    0238
 EC      D /INTEL/   000104    0221S   0225S   0245    0246
 ECOSF   D /INTEL/   000074    0221S   0225S   0232
 ES      D /INTEL/   000110    0221S   0225S   0245    0246
 ESINF   D /INTEL/   000100    0221S   0225S   0244    0250    0256
 ETA     D /INTEL/   000000    0221S   0225S   0261    0262    0274
 GAM2P   D /INTEL/   000114    0221S   0225S   0250    0253    0261    0262    0264
                               0265    0274
 I       I           001446    0269M   0270    0276M   0277
 INTERM  I EXTERNAL  000000    0230
 KEY     I ARGUMENT  000004    0216S   0229A   0267
 KROW    I ARGUMENT  000003    0216S   0229A   0270    0277
 L       D /INTEL/   000024    0221S   0225S   0256    0260    0274
 L2Z     D /INTEL/   000034    0221S   0225S   0248    0249    0250    0253    0254
                               0263
 LAMBDA  D /INTEL/   000020    0221S   0225S   0244
 O       D /COREL/   000060    0221S   0225S   0270M   0277
 RHO     D /INTEL/   000040    0221S   0225S   0233    0248    0249    0250    0253
                               0254    0256    0259    0263    0264    0265    0273
                               0274
 RHO3    D           001451    0221S   0233M   0249    0250    0253    0254    0255
                               0257    0274
 S2U     D           001455    0221S   0241M   0250    0256
 SGH     D           001461    0221S   0248M   0250    0257
 SIGMA   D /INTEL/   000014    0221S   0225S   0234    0235    0264    0265    0274
 SINU    D /INTEL/   000070    0221S   0225S   0238    0243    0246    0261
 SU3     D           001465    0221S   0242M   0261
 SUP     D           001471    0221S   0245M   0247    0264    0265
 SUPS2   D           001475    0221S   0247M   0248    0250    0256    0263
 TAU     D /INTEL/   000010    0221S   0225S   0234    0235    0264    0265    0274
 U       D /INTEL/   000060    0221S   0225S   0236    0237    0238    0239    0244
```

```
UL      D             001501   0221S  0244M  0248   0263
VAR     D //          000000   0221S  0225S  0270   0277M
XFR     R EXTERNAL    000000   0229
XI      D /INTEL/     000004   0221S  0225S  0261   0262   0274
Z       D             001505   0221S  0232M  0249   0252   0254   0255
Z4      D             001511   0221S  0239M  0242   0243   0245   0246
Z5      D             001515   0221S  0238M  0242   0243   0245   0246   0259M
                               0261   0262
Z6      D             001521   0221S  0237M  0240   0241   0246   0258M  0261
                               0262
Z7      D             001525   0221S  0236M  0240   0241   0245   0256M  0257M
                               0261   0262
Z8      D             001531   0221S  0235M  0240   0241   0242   0243   0247
                               0252M  0253   0255M  0256   0260   0273M  0274
Z9      D             001535   0221S  0234M  0240   0241   0242   0243   0247
                               0249M  0250   0254M  0255   0258   0260M  0261
                               0262   0263M  0264   0265
ZETA    D /INTEL/     000030   0221S  0225S  0231   0250   0252   0255   0256
                               0257   0258   0274
ZETA2   D             001541   0221S  0231M  0250   0254   0258

_100                  001217   0267   0273D
_150                  001347   0276   0277D
_20                   000042   0230D
_50                   001167   0269   0270D
```

```
(0280) C
(0281)       SUBROUTINE INTERM
(0282) C
(0283) C     CALCULATE INTERMEDIATE QUANTITIES IN ELEMENT TRANSFORMATIONS
(0284) C
(0285)       DOUBLE PRECISION XI,ETA,SIGMA,TAU,L,LAMBDA,ZETA,L2Z,RHO,C1,C2,
(0286)      + ECLON,U,COSU,SINU,ECOSF,ESINF,EC,ES,DRAD,MU,GAM2N,GAM2P,Z8,Z9
(0287)       COMMON /INTEL/ETA,XI,TAU,SIGMA,LAMBDA,L,ZETA,L2Z,RHO,C1,C2,
(0288)      + ECLON,U,COSU,SINU,ECOSF,ESINF,EC,ES,GAM2P/COEF/DRAD,MU,GAM2N
(0289) C
(0290)       ZETA=1-(XI*XI+ETA*ETA)/(2*L)
(0291)       L2Z=2*L*ZETA
(0292)       GAM2P=GAM2N/L2Z**4
(0293)       RHO=1-(SIGMA*SIGMA+TAU*TAU)/L2Z
(0294)       C1=DSQRT((1+ZETA)/(2*L))
(0295)       C2=DSQRT((1+RHO)/L2Z)
(0296)       EC=C1*XI
(0297)       ES=-C1*ETA
(0298)       ECLON=LAMBDA
(0299) 100   Z9=ECLON
(0300)       Z8=1-EC*DCOS(Z9)-ES*DSIN(Z9)
(0301)       ECLON=Z9-(Z9-LAMBDA-EC*DSIN(Z9)+ES*DCOS(Z9))/Z8
(0302)       IF(DABS(ECLON-Z9).GT.2.D-14*(DABS(Z9)+1))GOTO 100
(0303)       Z9=(ECLON-LAMBDA)
(0304)       Z8=Z8+ZETA
(0305)       U=ECLON+2*DATAN2(Z9,Z8)
(0306)       SINU=DSIN(U)
(0307)       COSU=DCOS(U)
(0308)       ESINF=EC*SINU-ES*COSU
(0309)       ECOSF=EC*COSU+ES*SINU
(0310)       RETURN
(0311)       END
```

C1	D	/INTEL/	000044	0285S	0287S	0294M	0296	0297		
C2	D	/INTEL/	000050	0285S	0287S	0295M				
COSU	D	/INTEL/	000064	0285S	0287S	0307M	0308	0309		
DABS	D	EXTERNAL	000000	0302						
DATAN2	D	EXTERNAL	000000	0305						
DCOS	D	EXTERNAL	000000	0300	0301	0307				
DSIN	D	EXTERNAL	000000	0300	0301	0306				
DSQRT	D	EXTERNAL	000000	0294	0295					
EC	D	/INTEL/	000104	0285S	0287S	0296M	0300	0301	0308	0309
ECLON	D	/INTEL/	000054	0285S	0287S	0298M	0299	0301M	0302	0303
				0305						
ECOSF	D	/INTEL/	000074	0285S	0287S	0309M				
ES	D	/INTEL/	000110	0285S	0287S	0297M	0300	0301	0308	0309
ESINF	D	/INTEL/	000100	0285S	0287S	0308M				
ETA	D	/INTEL/	000000	0285S	0287S	0290	0297			
GAM2N	D	/COEF/	000010	0285S	0287S	0292				
GAM2P	D	/INTEL/	000114	0285S	0287S	0292M				
L	D	/INTEL/	000024	0285S	0287S	0290	0291	0294		
L2Z	D	/INTEL/	000034	0285S	0287S	0291M	0292	0293	0295	

LAMBDA	D	/INTEL/	000020	0285S	0287S	0298	0301	0303		
RHO	D	/INTEL/	000040	0285S	0287S	0293M	0295			
SIGMA	D	/INTEL/	000014	0285S	0287S	0293				
SINU	D	/INTEL/	000070	0285S	0287S	0306M	0308	0309		
TAU	D	/INTEL/	000010	0285S	0287S	0293				
U	D	/INTEL/	000060	0285S	0287S	0305M	0306A	0307A		
XI	D	/INTEL/	000004	0285S	0287S	0290	0296			
Z8	D		000344	0285S	0300M	0301	0304M	0305A		
Z9	D		000350	0285S	0299M	0300A	0301A	0302A	0303M	0305A
ZETA	D	/INTEL/	000030	0285S	0287S	0290M	0291	0294	0304	
_100			000144	0299D	0302					

```
(0312) C
(0313)        SUBROUTINE XFR(KROW,KEY)
(0314) C
(0315) C      TRANSFER OSCULATING ELEMENTS O TO WORKING ELEMENTS W (KEY=1)
(0316) C      OR MEAN ELEMENTS VAR TO WORKING ELEMENTS (KEY=2) OR THE REVERSE
(0317) C      (KEY=3 OR 4).
(0318) C
(0319)        DOUBLE PRECISION C(6),E(6),O(6),VAR(6,17),W(6)
(0320)        COMMON /INTEL/W/COREL/C,E,O//VAR
(0321) C
(0322)        GOTO (10,20,30,40),KEY
(0323)  10    DO 15 I=1,6
(0324)  15    W(I)=O(I)
(0325)        RETURN
(0326)  20    DO 25 I=1,6
(0327)  25    W(I)=VAR(I,KROW)
(0328)        K9=KROW
(0329)        RETURN
(0330)  30    DO 35 I=1,6
(0331)  35    O(I)=W(I)
(0332)        RETURN
(0333)  40    DO 45 I=1,6
(0334)  45    VAR(I,KROW)=W(I)
(0335)  .     RETURN
(0336)        END
```

```
I      I              000134   0323M  0324   0326M  0327   0330M  0331   0333M
                               0334
K9     I              000136   0328M
KEY    I ARGUMENT     000004   0313S  0322
KROW   I ARGUMENT     000003   0313S  0327   0328   0334
O      D /COREL/      000060   0319S  0320S  0324   0331M
VAR    D //           000000   0319S  0320S  0327   0334M
W      D /INTEL/      000000   0319S  0320S  0324M  0327M  0331   0334

_10                   000014   0322   0323D
_15                   000016   0323   0324D
_20                   000034   0322   0326D
_25                   000036   0326   0327D
_30                   000065   0322   0330D
_35                   000067   0330   0331D
_40                   000105   0322   0333D
_45                   000107   0333   0334D
```

```
(0337)  C
(0338)        SUBROUTINE READ(KROW)
(0339)  C
(0340)  C     READ SUN AND MOON COORDINATES FROM DISK FILE
(0341)  C
(0342)        DOUBLE PRECISION TSL,SL(17,8),ZZ
(0343)        COMMON/LUNSOL/TSL,SL
(0344)        READ(5,100)TSL,(SL(KROW,I),I=5,7)
(0345)        READ(5,100)(SL(KROW,I),I=1,3)
(0346)        READ(5,100)ZZ
(0347)        READ(5,100)ZZ
(0348)  100   FORMAT(4D22.14)
(0349)        DO 200 I=1,2
(0350)        N=4*(I-1)
(0351)        SL(KROW,N+4)=DSQRT(SL(KROW,N+1)**2+SL(KROW,N+2)**2+SL(KROW,
(0352)       + N+3)**2)
(0353)        DO 150 J=1,3
(0354)  150   SL(KROW,N+J)=SL(KROW,N+J)/SL(KROW,N+4)
(0355)  200   CONTINUE
(0356)        RETURN
(0357)        END
```

```
DSQRT   D EXTERNAL 000000   0351
I       I          000231   0344M  0345M  0349M  0350
J       I          000234   0353M  0354
KROW    I ARGUMENT 000003   0338S  0344   0345   0351   0354
N       I          000235   0350M  0351   0354
SL      D /LUNSOL/ 000004   0342S  0343S  0344M  0345M  0351M  0354M
TSL     D /LUNSOL/ 000000   0342S  0343S  0344M
ZZ      D          000236   0342S  0346M  0347M

 100               000077   0344   0345   0346   0347   0348D
 150               000155   0353   0354D
 200               000211   0349   0355D
```

```
(0358)  C
(0359)        SUBROUTINE QUE(M,Q0,Y,NT,Q,J)
(0360)  C
(0361)  C          Q(JS,N)= Q M/N    :    Q(JL,N)= Q M+1/N
(0362)  C
(0363)        DOUBLE PRECISION Q0(2,20),Y,Q
(0364)        IF(M.GT.1) Q0(J,M-1)=0
(0365)        IF(M.GT.NT)RETURN
(0366)        IF(M.EQ.0)GOTO 100
(0367)        Q0(J,M)=Q
(0368)        IF(M.EQ.NT)RETURN
(0369)        Q0(J,M+1)=(2*M+1)*Y*Q
(0370)        N=M+2
(0371)        GOTO 200
(0372)  100   Q0(J,1)=Y
(0373)        Q0(J,2)=1.5*Y*Y-.5
(0374)        N=3
(0375)  150   Q0(J,N)=((2*N-1)*Y*Q0(J,N-1)-(N+M-1)*Q0(J,N-2))/(N-M)
(0376)        N=N+1
(0377)  200   IF(N.LE.NT)GOTO 150
(0378)        Q=(2*M+1)*Q
(0379)        RETURN
(0380)        END
```

```
 J       I ARGUMENT 000010   0359S  0364   0367   0369   0372   0373   0375
 M       I ARGUMENT 000003   0359S  0364   0365   0366   0367   0368   0369
                             0370   0375   0378
 N       I          000202   0370M  0374M  0375   0376M  0377
 NT      I ARGUMENT 000006   0359S  0365   0368   0377
 Q       D ARGUMENT 000007   0359S  0363S  0367   0369   0378M
 Q0      D ARGUMENT 000004   0359S  0363S  0364M  0367M  0369M  0372M  0373M
                             0375M
 Y       D ARGUMENT 000005   0359S  0363S  0369   0372   0373   0375

_100                000070   0366   0372D
_150                000112   0375D  0377
_200                000156   0371   0377D
```

```
C
      SUBROUTINE AYBEE(M,BO,X,NT,B,J,IDEC)
C
      DOUBLE PRECISION BO(2,20),B,X,X2,PO,P1
      X2=X*X
      IF(IDEC.EQ.1)GOTO 200
C
C          BO(JS,N)= A M/N+2    :   BO(JL,N)= A M+1/N+2
C
      PO=B
      P1=(2*M+1)*B/(M+1)
      B=-P1
      N=M+1
      IF(M.GT.2) BO(J,M-2)=PO
      IF(M.GT.1) BO(J,M-1)=P1
      GOTO 150
100   BO(J,N)=((2*N+1)*P1-(N+M)*(N-M)*PO/(X2*N))/(N+1)
      PO=P1
      P1=BO(J,N)
      N=N+1
150   IF(N.GT.NT)RETURN
      GOTO 100
C
C          BO(JS,N)= B M/N    :   BO(JL,N)= B M+1/N
C
200   IF(M.EQ.0) GOTO 210
      BO(J,M)=0
      IF(M.GT.1) BO(J,M-1)=0
      IF(M.GE.NT)RETURN
210   BO(J,M+1)=B
      IF(M.EQ.NT-1)RETURN
      BO(J,M+2)=(M+1)*X2*B
      N=M+3
      GOTO 250
220   BO(J,N)=(N-1)*X2/((N-M-1)*(N+M-1))*((2*N-3)*BO(J,N-1)-(N-2)
     + *BO(J,N-2))
      N=N+1
250   IF(N.LE.NT)GOTO 220
      B=B*X2/2
      RETURN
      END
```

B	D	ARGUMENT	000007	0382S	0384S	0390	0391	0392M	0410	0412
				0419M						
B0	D	ARGUMENT	000004	0382S	0384S	0394M	0395M	0397M	0399	0407M
				0408M	0410M	0412M	0415M			
IDEC	I	ARGUMENT	000011	0382S	0386					
J	I	ARGUMENT	000010	0382S	0394	0395	0397	0399	0407	0408
				0410	0412	0415				
M	I	ARGUMENT	000003	0382S	0391	0393	0394	0395	0397	0406
				0407	0408	0409	0410	0411	0412	0413
				0415						
N	I		000371	0393M	0397	0399	0400M	0401	0413M	0415
				0417M	0418					
NT	I	ARGUMENT	000006	0382S	0401	0409	0411	0418		
P0	D		000372	0384S	0390M	0394	0397	0398M		
P1	D		000376	0384S	0391M	0392	0395	0397	0398	0399M
X	D	ARGUMENT	000005	0382S	0384S	0385				
X2	D		000402	0384S	0385M	0397	0412	0415	0419	
_100			000106	0397D	0402					
_150			000172	0396	0401D					
_200			000177	0386	0406D					
_210			000227	0406	0410D					
_220			000263	0415D	0418					
_250			000344	0414	0418D					

TCOR

By a call to XFR, moves the osculating elements into the working area. Call INTERM to get the intermediate quantities appropriate to those elements. Calculate both elliptic elements and Cartesian coordinates.

TRAN (KROW, KEY)

KROW points to the step value in VAR to which the mean elements apply.

KEY = 1

Call XFR and INTERM to move the osculating elements into position. Calculate the short-period terms that are the differences between mean and osculating elements. Apply these and store the mean elements obtained in the propér place in VAR.

NOTE: In this process, only *part* of the second-order short-period correction to L is made, the remainder being done more conveniently in DERIV.

KEY = 2

Execute the reverse process, from mean elements to osculating elements.

TERM

Using the working elements as a base, calculate the intermediate quantities (e.g., ZETA, ESINE, EC). (Here also Kepler's equation is solved).

R

Moves osculating elements or mean elements into working area or reverse.

AD

Reads the file LUNSOL to acquire lunar solar coordinates and calculates the unit vector and the distance to each body.

GPS

1. The program starts with DATA statements defining the numerical constants and the initial parameters of the problem, which include the initial Cartesian coordinates or elliptic elements for the satellite in the last of the DATA statements (line 29 of the listings).

2. After initialization and calculation of the starting values of the mean elements and their time derivatives, the program in line 57 begins the cycling to set up the starting solution.

3. When this is complete, lines 81 through 89 allow printing information about the starting solution. Note that all of these lines can be commented out or removed without affecting the operation of the program. See also the note below in paragraph 5.

4. Lines 93 through 107 then apply one predictor and one corrector step to move the solution forward one time step; line 116 governs the repetitions of this.

5. Lines 108 through 115 allow printing information about the continuing solution, again these lines can be changed or removed.

   NOTE: The form of the write statements in this program, as well as that of the READ statements in subroutine READ (lines 344-347), might need to be altered for running the program on a different computer. PRIME Fortran IV *(in which the program is written)* recognizes logical unit 1 as the terminal and logical unit 5 as the first file channel. The proper file for reading is opened at the operating system level by OPEN LUNSOL 1 1, which opens the disk file LUNSOL on file channel 1 for reading only.

ENTER (KEY)

1. KEY = 1

   Transforms Cartesian coordinates (position and velocity) into osculating elements and, by a call to INTERM, prepares the intermediate quantities useful in further transformations. Calls XFR to move the working set of osculating elements to safe storage.

2. KEY = 2

   Transforms elliptic elements into osculating elements (calculating the intermediate quantities in the process) and, by calling XFR, moves the working elements into safe storage as the osculating elements.

PREVIOUS PAGE IS BLANK

APPENDIX B

PROGRAM AND SUBROUTINES

```
(0656)  C
(0657)        SUBROUTINE EXTRAP(KROW1,KROW)
(0658)  C
(0659)        DOUBLE PRECISION DEL(18,6,17),ADAM(17),VAR(6,17),VEL(6,17),
(0660)       + S,XX(6),HH(20)
(0661)        DIMENSION NT(3)
(0662)        COMMON //VAR,VEL,DEL,NEQ,KMAX/COEF/XX,NT,HH,ADAM
(0663)        DO 110 N=1,NEQ
(0664)        S=0
(0665)        DO 100 KC=1,KMAX
(0666)        KC1=KMAX+1-KC
(0667)        DEL(KC1,N,KROW)=DEL(KC1,N,KROW1)+DEL(KC1+1,N,KROW)
(0668)  100   S=S+ADAM(KC1)*DEL(KC1,N,KROW)
(0669)  110   VAR(N,KROW)=VAR(N,KROW1)+S
(0670)        CALL DERIV(KROW,1)
(0671)        RETURN
(0672)        END
```

```
ADAM   D /COEF/     000153   0659S   0662S   0668
DEL    D //         001460   0659S   0662S   0667M   0668
DERIV  R EXTERNAL   000000   0670
KC     I            000146   0665M   0666
KC1    I            000147   0666M   0667    0668
KMAX   I //         017741   0662S   0665    0666
KROW   I ARGUMENT   000004   0657S   0667    0668    0669    0670A
KROW1  I ARGUMENT   000003   0657S   0667    0669
N      I            000150   0663M   0667    0668    0669
NEQ    I //         017740   0662S   0663
S      D            000151   0659S   0664M   0668M   0669
VAR    D //         000000   0659S   0662S   0669M

 100                000053   0665    0668D
 110                000106   0663    0669D
```

100	002411	0484	0510D		
¯110	002537	0517	0518	0519D	
¯120	002662	0533	0535D		
¯130	002702	0529	0537D		
¯140	002752	0542	0543D		
¯150	002771	0536	0544D		
¯160	003037	0551	0553D		
¯180	003115	0557	0562D		
¯200	003175	0561	0569D		
¯300	003246	0573	0576D		
¯400	003452	0577	0588	0590D	
¯410	003525	0594	0596D		
¯500	003571	0555	0576	0598D	
¯510	003577	0597	0599D		
¯530	004165	0599	0617D		
¯540	004171	0617	0618	0619D	
¯550	004350	0628	0630D		
¯560	004401	0623	0632D		
¯570	004547	0641	0643D		
¯600	004572	0443	0620	0631	0644D
¯650	004637	0645	0648D		
¯70	001170	0438	0440D		
¯80	001434	0465	0466	0467D	
¯90	002006	0489	0490	0491D	

QUE	R	EXTERNAL	000000	0558	0570	0627	0640			
R	D		005027	0428S	0622M	0624	0625	0632	0635	
RE	D	/COEF/	000014	0428S	0434S	0531	0624			
RHO	D	/INTEL/	000040	0428S	0434S	0451	0452	0458	0459	0468
				0483	0503	0504	0509	0603	0612	0636
RHO2	D		005033	0428S	0603M	0605	0606	0608	0609	0612
SIGMA	D	/INTEL/	000014	0428S	0434S	0449	0450	0454	0455	0460
				0462	0471	0485	0494	0498	0507	0621
				0637						
SINB	D		005037	0428S	0621M	0627A	0636	0637		
SINU	D	/INTEL/	000070	0428S	0434S	0621	0637			
SL	D	/LUNSOL/	000004	0428S	0434S	0485	0486	0487	0488	0491
				0493	0494	0495	0496	0498	0499	0500
				0501	0507	0508	0537	0632	0637	
SM	D		005043	0428S	0550M	0566	0568M	0597		
SM1	D		005047	0428S	0566M	0567	0568	0592		
TAU	D	/INTEL/	000010	0428S	0434S	0449	0450	0453	0456	0461
				0463	0472	0485	0493	0499	0508	0621
				0637						
V	D		005053	0428S	0609M	0614	0615	0616		
V1	D		005057	0428S	0449M	0451	0481	0486	0493	0494
				0495	0496	0497				
V2	D		005063	0428S	0450M	0458	0482	0487	0498	0499
				0500	0501	0502				
VAR	D	//	000000	0428S	0434S	0647M				
VEL	D	//	000630	0428S	0434S	0648M	0649M	0650M	0651M	0652M
				0653M						
X	D		005067	0428S	0527M	0548	0559A	0571A	0585	
XFR	R	EXTERNAL	000000	0441						
XI	D	/INTEL/	000004	0428S	0434S	0449	0450	0453	0456	0460
				0462	0469	0475	0486	0487	0493	0496
				0498	0500	0505	0511	0514		
Z5	D		005073	0428S	0608M	0609	0610	0611		
Z6	D		005077	0428S	0607M	0609	0612			
Z7	D		005103	0428S	0485M	0486	0487	0488	0492M	0493
				0494	0495	0496	0497	0498	0499	0500
				0501	0502	0503	0538M	0539M	0540	0606M
				0609	0614	0625M	0629M	0630	0633M	0634M
				0635M	0642M	0643				
Z8	D		005107	0428S	0451M	0452	0455	0456	0458M	0459
				0462	0463	0504M	0505	0506	0532M	0534M
				0535	0540M	0541	0582M	0583	0586	0589
				0605M	0609	0614	0624M	0625	0629	0632M
				0635	0642					
Z9	D		005113	0428S	0452M	0453	0454	0457	0459M	0460
				0461	0464	0468M	0469	0470	0473	0474M
				0475	0476	0479	0503M	0504	0507	0508
				0509	0510M	0511	0512	0513	0514	0515
				0516	0531M	0532	0534	0537M	0540	0541
				0543	0581M	0582	0585	0604M	0609	0610
				0611	0612	0614	0636M	0637M	0640A	
ZETA	D	/INTEL/	000030	0428S	0434S	0474	0479	0510	0513	0516
				0527	0605	0609	0612	0614	0622	

ECOSF	D	/INTEL/	000074	0428S	0434S	0622				
ETA	D	/INTEL/	000000	0428S	0434S	0449	0450	0454	0455	0461
				0463	0470	0476	0486	0487	0494	0495
				0499	0501	0506	0512	0515		
FO	D		000546	0428S	0552M	0583M	0584M	0585M	0586M	0591
				0592	0593M	0595	0596M			
GAM2P	D	/INTEL/	000114	0428S	0434S	0604				
GAMMA	D		004775	0428S	0483M	0488M	0558A	0570A		
H	D		000576	0428S	0535M	0541M	0543M	0581	0584	
HO	D	/COEF/	000033	0428S	0434S	0535	0630			
I	I		005001	0438M	0439	0440	0466M	0467	0490M	0491
				0518M	0519	0551M	0552	0553	0594M	0595
				0596	0617M	0619				
INTERM	I	EXTERNAL	000000	0442						
ISUM	I		005005	0443M	0448	0484	0526	0529	0539	0548
				0559A	0571A	0579	0589	0599	0623	0634
J	I		005006	0465M	0467	0489M	0491	0517M	0519	0618M
				0619						
JL	I		005007	0569M	0570A	0571A	0584	0585		
JS	I		005010	0545M	0562M	0569	0580	0581	0582	0584
				0585						
KEY	I	ARGUMENT	000004	0423S	0588	0616	0620	0645		
KR	I	ARGUMENT	000003	0423S	0441A	0485	0486	0487	0488	0491
				0493	0494	0495	0496	0498	0499	0500
				0501	0507	0508	0537	0632	0637	0647
				0648	0649	0650	0651	0652	0653	
KS	I		005011	0448M	0485	0486	0487	0488	0491	0493
				0494	0495	0496	0498	0499	0500	0501
				0507	0508	0537	0632	0637		
L	D	/INTEL/	000024	0428S	0434S	0474	0480	0510	0528	0604
				0615	0646M	0647	0652			
L2Z	D	/INTEL/	000034	0428S	0434S	0449	0450	0468	0471	0472
				0481	0482	0492	0604			
M	I		005013	0556M	0557	0563	0570	0571	0572	0573
				0574	0575	0585	0587	0591	0592	
MM	D	/COEF/	000020	0428S	0434S	0538	0633			
MS	D	/COEF/	000024	0428S	0434S	0539	0634			
MU	D	/COEF/	000004	0428S	0434S	0480	0528	0532	0540	0604
				0615	0625	0635	0646	0652		
MUP	I		005014	0555M	0556					
N	I		005015	0533M	0535	0542M	0543	0577M	0578	0579
				0581	0582	0584	0585	0587	0589	0628M
				0630	0641M	0643				
NO	I		005016	0572M	0574M	0576	0577	0597		
N5	I		005017	0578M	0579M	0585	0586			
NMU	I		005020	0554M	0555					
NN	I		005021	0580M						
NTER	I		005022	0526M	0533	0542	0554	0558A	0559A	0570A
				0571A	0576	0577	0627A	0628	0640A	0641
NTERM	I	/COEF/	000030	0433S	0434S	0526				
Q	D		000716	0428S	0558A	0570A	0581	0584	0627A	0630
				0640A	0643					
Q1	D		005023	0428S	0544M	0558A	0570A	0626M	0627A	0639M
				0640A						

```
(0634)          IF (ISUM.EQ.3) Z7=MS
(0635)          Z7=Z7*MU*Z8*Z8/R
(0636)          Z9=C2*SINB/(1+RHO)
(0637)          Z9=SL(KR,KS+1)*(COSU-Z9*TAU)+SL(KR,KS+2)*(SINU-Z9*SIGMA)+
(0638)         + SL(KR,KS+3)*SINB
(0639)          Q1=1
(0640)          CALL QUE(0,Q,Z9,NTER,Q1,1)
(0641)          DO 570 N=2,NTER
(0642)          Z7=Z7*Z8
(0643)    570   DELF2=DELF2-Z7*Q(1,N)
(0644)    600   CONTINUE
(0645)          IF (KEY.EQ.1) GOTO 650
(0646)          L=L+L**3/(MU*MU)*DELF2
(0647)          VAR(6,KR)=L
(0648)    650   VEL(1,KR)=-DE(2)
(0649)          VEL(2,KR)=DE(1)
(0650)          VEL(3,KR)=-DE(4)
(0651)          VEL(4,KR)=DE(3)
(0652)          VEL(5,KR)=MU*MU/L**3-DE(5)
(0653)          VEL(6,KR)=0
(0654)          RETURN
(0655)          END
```

A	D		004715	0428S	0528M	0531	0532	0537	0540	0586
				0622						
ALPHA	D		004721	0428S	0481M	0486M	0511	0512	0513	0520M
				0567	0568	0607	0610			
AYBEE	R	EXTERNAL	000000	0559	0571					
B	D		000006	0428S	0559A	0571A	0582	0584	0585	
B1	D		004725	0428S	0547M	0548M	0559A	0571A		
BETA	D		004731	0428S	0482M	0487M	0514	0515	0516	0521M
				0567	0568	0607	0611			
C	D		000246	0428S	0439M	0453M	0454M	0455M	0456M	0457M
				0460M	0461M	0462M	0463M	0464M	0467M	0469M
				0470M	0471M	0472M	0473M	0475M	0476M	0477M
				0478M	0479M	0480M	0491M	0493M	0494M	0495M
				0496M	0497M	0498M	0499M	0500M	0501M	0502M
				0505M	0506M	0507M	0508M	0509M	0511M	0512M
				0513M	0514M	0515M	0516M	0519M	0619	
C1	D	/INTEL/	000044	0428S	0434S	0519	0520	0521		
C2	D	/INTEL/	000050	0428S	0434S	0467	0481	0482	0488	0505
				0506	0507	0508	0509	0621	0636	
CM	D		004735	0428S	0549M	0565	0567M	0589	0595	0597
CM1	D		004741	0428S	0565M	0567	0568	0591		
COSU	D	/INTEL/	000064	0428S	0434S	0621	0637			
D	D		004745	0428S	0560M	0564M	0575M	0581	0584	0587M
D1	D		004765	0428S	0546M	0560	0563M	0564		
DE	D		000466	0428S	0440M	0619M	0648	0649	0650	0651
				0652						
DELF2	D		004771	0428S	0530M	0589M	0616M	0630M	0643M	0646
DF	D		000516	0428S	0553M	0591M	0592M	0595M	0610M	0611M
				0612M	0614M	0615M	0619			

```
      Z9=D*H(N)*Q(JS,N)
      Z8=Z9*B(JS,N)
      FO(1)=FO(1)+Z8
      FO(3)=FO(3)+D*H(N)*Q(JL,N)*B(JS,N)
      FO(4)=FO(4)+(M-N5-1)/X*Z9*(B(JS,N)+B(JL,N))
      FO(5)=FO(5)+N5/A*Z8
      D=-D*(N-M+1)/(N+M+2)
      IF (KEY.EQ.1) GOTO 400
      IF (N.GT.2.OR.ISUM.GT.1) DELF2=DELF2+Z8*CM
  400 CONTINUE
      DF(1)=DF(1)+FO(1)*M*CM1
      DF(2)=DF(2)-FO(1)*M*SM1
      FO(1)=0
      DO 410 I=3,5
      DF(I)=DF(I)+FO(I)*CM
  410 FO(I)=0
      IF(CM*CM+SM*SM.LT.1.0D-8.AND.NO.GT.2)GOTO 510
  500 CONTINUE
  510 IF (ISUM.NE.1)GOTO 530
C
C     SECOND ORDER CONTRIBUTION OF SECOND HARMONIC
C
      RHO2=RHO*RHO
      Z9=(MU*GAM2P)**2*L2Z/(48*L**3)
      Z8=5*ZETA*(1+RHO2*(-3.6+RHO2))
      Z7=4*(1+RHO2*(-6+9*RHO2))
      Z6=ALPHA*ALPHA-BETA*BETA
      Z5=2-30*RHO2
      V=Z9*((Z8+Z7)*ZETA+5*(-1+RHO2*(2+7*RHO2))-Z6*Z5)
      DF(1)=DF(1)-2*Z9*Z5*ALPHA
      DF(2)=DF(2)+2*Z9*Z5*BETA
      DF(3)=DF(3)+Z9*4*RHO*(5*(1+7*RHO2)+ZETA*(12*(-1+3*RHO2)+ZETA*
     + (-9+5*RHO2))+15*Z6)
      DF(4)=DF(4)-ZETA*(ZETA*Z9*(2*Z8+Z7)-7*V)
      DF(5)=DF(5)-5*MU*V/(L*L)
      IF (KEY.NE.1) DELF2=DELF2+V
  530 DO 540 I=1,5
      DO 540 J=1,5
  540 DE(I)=DE(I)+DF(J)*C(J,I)
      IF (KEY.EQ.1) GOTO 600
      SINB=C2*(SIGMA*SINU+TAU*COSU)
      R=A*ZETA*ZETA/(ECOSF+1)
      IF (ISUM.NE.1) GOTO 560
      Z8=RE/R
      Z7=-MU/R*Z8*Z8
      Q1=1
      CALL QUE(0,Q,SINB,NTER,Q1,1)
      DO 550 N=3,NTER
      Z7=Z7*Z8
  550 DELF2=DELF2-Z7*HO(N)*Q(1,N)
      GOTO 600
  560 Z8=R/SL(KR,KS+4)
      Z7=MM
```

```
       A=L*L/MU
       IF (ISUM.NE.1)GOTO 130
       DELF2=0
       Z9=RE/A
       Z8=MU/A*Z9
       DO 120 N=2,NTER
       Z8=Z8*Z9
  120  H(N)=-H0(N)*Z8
       GOTO 150
  130  Z9=A/SL(KR,KS+4)
       Z7=MM
       IF(ISUM.EQ.3)Z7=MS
       Z8=MU*Z7/A*Z9
       H(1)=Z8*Z9
       DO 140 N=2,NTER
  140  H(N)=H(N-1)*Z9
  150  Q1=1
       JS=1
       D1=2
       B1=1
       IF(ISUM.EQ.1)B1=X
       CM=1
       SM=0
       DO 160 I=1,6
       F0(I)=0
  160  DF(I)=0
       NMU=NTER+1
       DO 500 MUP=1,NMU
       M=MUP-1
       IF (M.GT.0)GOTO 180
       CALL QUE(0,Q,GAMMA,NTER,Q1,1)
       CALL AYBEE(0,B,X,NTER,B1,1,ISUM)
       D=D1/2
       GOTO 200
  180  JS=3-JS
       D1=D1/(2*M)
       D=D1
       CM1=CM
       SM1=SM
       CM=CM1*ALPHA-SM1*BETA
       SM=SM1*ALPHA+CM1*BETA
  200  JL=3-JS
       CALL QUE(M+1,Q,GAMMA,NTER,Q1,JL)
       CALL AYBEE(M+1,B,X,NTER,B1,JL,ISUM)
       N0=M
       IF (M.GE.2) GOTO 300
       N0=M+2
       D=-D/(2*M+2)
  300  IF(N0.GT.NTER)GOTO 500
       DO 400 N=N0,NTER,2
       N5=N
       IF(ISUM.EQ.1)N5=-N-1
       NN=JS
```

```
      C(4,2)=-XI*Z9
      C(4,1)=-ETA*Z9
      C(4,4)=0
      C(4,3)=0
      C(4,5)=(1-ZETA)*Z9
      C(5,5)=2*L/MU
      ALPHA=C2*V1*L2Z
      BETA=C2*V2*L2Z
      GAMMA=RHO
      IF(ISUM.EQ.1)GOTO 100
      Z7=SL(KR,KS+1)*TAU+SL(KR,KS+2)*SIGMA
      ALPHA=SL(KR,KS+3)*ALPHA+SL(KR,KS+1)*XI-SL(KR,KS+2)*ETA-V1*Z7
      BETA=SL(KR,KS+3)*BETA-SL(KR,KS+1)*ETA-SL(KR,KS+2)*XI-V2*Z7
      GAMMA=SL(KR,KS+3)*GAMMA-C2*Z7
      DO 90 J=1,5
      DO 90 I=1,3
90    C(I,J)=SL(KR,KS+3)*C(I,J)
      Z7=Z7/L2Z
      C(1,2)=C(1,2)+SL(KR,KS+1)-Z7*(TAU+2*V1*XI)
      C(1,1)=C(1,1)-SL(KR,KS+2)+Z7*(SIGMA-2*V1*ETA)
      C(1,4)=C(1,4)-SL(KR,KS+2)*V1+ETA*Z7
      C(1,3)=C(1,3)-SL(KR,KS+1)*V1-XI*Z7
      C(1,5)=C(1,5)+2*V1*Z7
      C(2,2)=C(2,2)-SL(KR,KS+2)+Z7*(SIGMA-2*V2*XI)
      C(2,1)=C(2,1)-SL(KR,KS+1)+Z7*(TAU-2*V2*ETA)
      C(2,4)=C(2,4)-SL(KR,KS+2)*V2+XI*Z7
      C(2,3)=C(2,3)-SL(KR,KS+1)*V2+ETA*Z7
      C(2,5)=C(2,5)+2*V2*Z7
      Z9=Z7/(1+RHO)
      Z8=2*RHO*Z9
      C(3,2)=C(3,2)-C2*XI*Z8
      C(3,1)=C(3,1)-C2*ETA*Z8
      C(3,4)=C(3,4)-C2*(SL(KR,KS+2)-SIGMA*Z9)
      C(3,3)=C(3,3)-C2*(SL(KR,KS+1)-TAU*Z9)
      C(3,5)=C(3,5)+C2*2*RHO*Z9
100   Z9=1/(2*L*(1+ZETA))
      C(1,2)=C(1,2)-ALPHA*XI*Z9
      C(1,1)=C(1,1)-ALPHA*ETA*Z9
      C(1,5)=C(1,5)-2*ALPHA*ZETA*Z9
      C(2,2)=C(2,2)-BETA*XI*Z9
      C(2,1)=C(2,1)-BETA*ETA*Z9
      C(2,5)=C(2,5)-2*BETA*ZETA*Z9
      DO 110 J=1,5
      DO 110 I=1,2
110   C(I,J)=C1*C(I,J)
      ALPHA=C1*ALPHA
      BETA=C1*BETA
C
C     RECURSIVE CALCULATION OF DERIVATIVES OF PERTURBING
C     POTENTIAL WITH RESPECT TO XI,ETA,SIGMA,TAU,L
C
      NTER=NTERM(ISUM)
      X=1/ZETA
```

```
C
      SUBROUTINE DERIV(KR,KEY)
C
C     KEY=1:  DO NOT CALCULATE SECOND ORDER CORRECTIONS TO L
C     KEY=2:  DO
C
      DOUBLE PRECISION VAR(6,17),VEL(6,17),XI,ETA,SIGMA,TAU,L,LAMBDA,
     + ZETA,L2Z,RHO,C1,C2,V1,V2,Z5,Z6,Z7,Z8,Z9,ALPHA,BETA,GAMMA,C(6,6),
     + DRAD,MU,GAM2N,RE,MM,MS,X,A,B1,D,D1,Q1,CM1,SM1,CM,SM,R,B(2,20),
     + Q(2,20),F0(6),DF(6),DE(6),SL(17,8),H(20),H0(20),TT,DELF2,SINB,
     + ECLON,U,SINU,COSU,ECOSF,ESINF,EC,ES,GAM2P,RHO2,V
      DIMENSION NTERM(3)
      COMMON/INTEL/ETA,XI,TAU,SIGMA,LAMBDA,L,ZETA,L2Z,RHO,C1,C2,
     + ECLON,U,COSU,SINU,ECOSF,ESINF,EC,ES,GAM2P
     + //VAR,VEL/COEF/DRAD,MU,GAM2N,RE,MM,MS,NTERM,H0/LUNSOL/TT,SL
C
      DO 70 I=1,5
      C(5,I)=0
   70 DE(I)=0
      CALL XFR(KR,2)
      CALL INTERM
      DO 600 ISUM=1,3         /*  1=ZONALS, 2=MOON, 3=SUN
C
C     JACOBIAN OF ALPHA,BETA,GAMMA,X,A WITH RESPECT TO
C     ETA,XI,TAU,SIGMA,L
C
      KS=4*(ISUM-2)
      V1=(XI*TAU-ETA*SIGMA)/L2Z
      V2=(-XI*SIGMA-ETA*TAU)/L2Z
      Z8=V1/(1+RHO)
      Z9=2*RHO*Z8
      C(1,2)=TAU+Z9*XI
      C(1,1)=-SIGMA+Z9*ETA
      C(1,4)=-ETA-SIGMA*Z8
      C(1,3)=XI-TAU*Z8
      C(1,5)=-Z9
      Z8=V2/(1+RHO)
      Z9=2*RHO*Z8
      C(2,2)=-SIGMA+Z9*XI
      C(2,1)=-TAU+Z9*ETA
      C(2,4)=-XI-SIGMA*Z8
      C(2,3)=-ETA-TAU*Z8
      C(2,5)=-Z9
      DO 80 J=1,5
      DO 80 I=1,2
   80 C(I,J)=C2*C(I,J)
      Z9=2*(1-RHO)/L2Z
      C(3,2)=-XI*Z9
      C(3,1)=-ETA*Z9
      C(3,4)=-2*SIGMA/L2Z
      C(3,3)=-2*TAU/L2Z
      C(3,5)=Z9
      Z9=-1/(L*ZETA*ZETA)
```

B	D	ARGUMENT	000007	0382S	0384S	0390	0391	0392M	0410	0412
				0419M						
BO	D	ARGUMENT	000004	0382S	0384S	0394M	0395M	0397M	0399	0407M
				0408M	0410M	0412M	0415M			
IDEC	I	ARGUMENT	000011	0382S	0386					
J	I	ARGUMENT	000010	0382S	0394	0395	0397	0399	0407	0408
				0410	0412	0415				
M	I	ARGUMENT	000003	0382S	0391	0393	0394	0395	0397	0406
				0407	0408	0409	0410	0411	0412	0413
				0415						
N	I		000371	0393M	0397	0399	0400M	0401	0413M	0415
				0417M	0418					
NT	I	ARGUMENT	000006	0382S	0401	0409	0411	0418		
P0	D		000372	0384S	0390M	0394	0397	0398M		
P1	D		000376	0384S	0391M	0392	0395	0397	0398	0399M
X	D	ARGUMENT	000005	0382S	0384S	0385				
X2	D		000402	0384S	0385M	0397	0412	0415	0419	
100			000106	0397D	0402					
150			000172	0396	0401D					
200			000177	0386	0406D					
210			000227	0406	0410D					
220			000263	0415D	0418					
250			000344	0414	0418D					

```
C
      SUBROUTINE DERIV(KR,KEY)
C
C     KEY=1:  DO NOT CALCULATE SECOND ORDER CORRECTIONS TO L
C     KEY=2:  DO
C
      DOUBLE PRECISION VAR(6,17),VEL(6,17),XI,ETA,SIGMA,TAU,L,LAMBDA,
     + ZETA,L2Z,RHO,C1,C2,V1,V2,Z5,Z6,Z7,Z8,Z9,ALPHA,BETA,GAMMA,C(6,6),
     + DRAD,MU,GAM2N,RE,MM,MS,X,A,B1,D,D1,Q1,CM1,SM1,CM,SM,R,B(2,20),
     + Q(2,20),F0(6),DF(6),DE(6),SL(17,8),H(20),H0(20),TT,DELF2,SINB,
     + ECLON,U,SINU,COSU,ECOSF,ESINF,EC,ES,GAM2P,RHO2,V
      DIMENSION NTERM(3)
      COMMON/INTEL/ETA,XI,TAU,SIGMA,LAMBDA,L,ZETA,L2Z,RHO,C1,C2,
     + ECLON,U,COSU,SINU,ECOSF,ESINF,EC,ES,GAM2P
     + //VAR,VEL/COEF/DRAD,MU,GAM2N,RE,MM,MS,NTERM,H0/LUNSOL/TT,SL
C
      DO 70 I=1,5
      C(5,I)=0
   70 DE(I)=0
      CALL XFR(KR,2)
      CALL INTERM
      DO 600 ISUM=1,3          /*  1=ZONALS, 2=MOON, 3=SUN
C
C     JACOBIAN OF ALPHA,BETA,GAMMA,X,A WITH RESPECT TO
C     ETA,XI,TAU,SIGMA,L
C
      KS=4*(ISUM-2)
      V1=(XI*TAU-ETA*SIGMA)/L2Z
      V2=(-XI*SIGMA-ETA*TAU)/L2Z
      Z8=V1/(1+RHO)
      Z9=2*RHO*Z8
      C(1,2)=TAU+Z9*XI
      C(1,1)=-SIGMA+Z9*ETA
      C(1,4)=-ETA-SIGMA*Z8
      C(1,3)=XI-TAU*Z8
      C(1,5)=-Z9
      Z8=V2/(1+RHO)
      Z9=2*RHO*Z8
      C(2,2)=-SIGMA+Z9*XI
      C(2,1)=-TAU+Z9*ETA
      C(2,4)=-XI-SIGMA*Z8
      C(2,3)=-ETA-TAU*Z8
      C(2,5)=-Z9
      DO 80 J=1,5
      DO 80 I=1,2
   80 C(I,J)=C2*C(I,J)
      Z9=2*(1-RHO)/L2Z
      C(3,2)=-XI*Z9
      C(3,1)=-ETA*Z9
      C(3,4)=-2*SIGMA/L2Z
      C(3,3)=-2*TAU/L2Z
      C(3,5)=Z9
      Z9=-1/(L*ZETA*ZETA)
```

QUE

Calculates the quantities Q required for the harmonics.

AYBEE

Calculates the quantities A and B required for the harmonics.

DERIV (KEY)

Loads the mean elements with XFR and INTERM; and in separate calculations for zonals, moon, and sun:

1. Finds the Jacobian matrix of the variables ALPHA, BETA, GAMMA, X, A (in terms of which the derivatives of the Hamiltonian are naturally determined) with respect to the mean elements.

2. Calculates the derivatives of the perturbing potential as well as the potential value itself. The second order of the second harmonic is included.

3. If KEY=2 (at entry to the program only), finds the instantaneous value of the perturbing potential and uses it to calculate the remaining part of the second-order-short-period correction for L. If KEY=1, this step is omitted.

4. Use the Jacobian and the derivatives to accumulate the derivatives of the mean elements required for the next integration step, storing these in VEL.

EXTRAP (KROW1, DROW)

Extends the difference table DEL from the preceding row KROW1 to the current row KROW, calculates the step in VAR, and calls DERIV for the new row.

SUBROUTINE TAXONOMY

NAME	CALLS	IS CALLED BY	USES COMMON
GPS (MAIN)	ENTER OSTCOR ELTRAN READ DERIV EXTRAP		(BLANK) COEF COREL
ENTER	INTERM XFR	GPS	COEF COREL INTEL
OSTCOR	INTERM XFR	GPS	COEF COREL
ELTRAN	INTERM XFR	GPS	(BLANK) COEF COREL INTEL
INTERM		ENTER OSTCOR ELTRAN	COEF INTEL
XFR		ENTER OSTCOR ELTRAN	(BLANK) COREL INTEL
READ		GPS	LUNSOL
QUE		DERIV	
AYBEE		DERIV	
DERIV	INTERM XFR QUE AYBEE	GPS EXTRAP	COEF INTEL LUNSOL
EXTRAP	DERIV	GPS	(BLANK) COEF

VARIABLES AND COMMON BLOCKS

(BLANK)	Integration quantities	
	VAR (6,17)	The six integration variables (mean elements) for 17 consecutive time steps
	VEL (6,17)	The derivatives in a similar array
	DEL (18,6,17)	The differences used in the numerical integration
	NEQ	Number of equations
	KMAX	Order of numerical integration procedure
COEF	Numerical coefficients	
	DRAD	Degrees/radians
	MU	Gravitational const
	GAM2N	12*J2*(MU*RE)**2
	RE	Earth radius
	MM	Lunar mass
	MS	Solar mass
	NTERM (3)	Highest order harmonic: zonal, lunar, solar
	HAR(20)	Coefficients of zonal harmonics
	ADAM (17)	Adams coefficients
COREL	Coordinates and elements	
	X,Y,Z,VX,VY,VZ (or C(6))	Coordinates
	A,ECC,INC,ELL,G,H (or E(6))	Elliptic elements
	O(6)	Osculating elements
INTEL	Intermediate quantities in element transformations	
	ETA,XI,TAU,SIGMA,LAMDA,L (or W(6))	Working elements (osculating on mean)

	[14 variables, such as ZETA, RHO, U, ECOSF— see listing]	Intermediate variables
LUNSOL	Lunar and solar coordinates	
	TSL	Time
	SL (17, 8)	For 17 time steps, the three components of the unit vector to the moon and the lunar distance; followed by the corresponding 4 quantities for the sun

The variables not found in common declarations are few in number and usually nonce variables. An exception is the variable KS that counts the steps in the integration. The variable TIM (which is defined but not used) is the time; its definition is

TIM=STEP*(KS-2)+TIMO (seconds)

External procedures not in the subroutine list (system subroutines or functions):

EXIT	DSQRT
MOD	DABS
SQRT	DATAN2
DCOS	DMOD
DSIN	DSIGN

All of the DOUBLE PRECISION declarations and functions are to produce a 64-bit value (48-bit mantissa). On a machine with sufficient word length, all the DOUBLE PRECISION declarations could be replaced with REAL and all the double functions with their single-precision counterparts.

APPENDIX C

COORDINATE TRANSFORMATIONS

TRANSFORMATION FROM CARTESIAN COORDINATES TO OSCULATING CANONICAL VARIABLES

[This is merely summarized here, since most of it is to be found in any text on orbital mechanics and the rest is obvious from the definition of the canonical variables. In the program, it is found in subroutine ENTER (lines 147-178)]. Vectors are defined by an underline _; unit vectors by a caret ^.

Angular momentum $\underline{h} = \underline{r} \times \underline{v}$

magnitude $h = \sqrt{h_x^2 + h_y^2 + h_z^2}$

Unit vector $$\hat{h} = \begin{pmatrix} \sin i \sin h \\ -\sin i \cos h \\ \cos i \end{pmatrix}$$

Hence, $$\hat{h} = \begin{pmatrix} -I_S \\ -I_C \\ \rho \end{pmatrix}$$

Unit vector $$\hat{L}_1 = \begin{pmatrix} 1 - I_S^2/(1+\rho) \\ -I_S I_C/(1+\rho) \\ I_S \end{pmatrix}$$

Unit vector $$\hat{L}_2 = \begin{pmatrix} -I_S I_C/(1+\rho) \\ 1 - I_C^2/(1+\rho) \\ I_C \end{pmatrix}$$

$\hat{L}_1$, $\hat{L}_2$, and $\hat{h}$ are mutually orthogonal and define a Cartesian system rotated from the XYZ system by the angle i about the line of nodes. Therefore, $\hat{L}_1$ and $\hat{L}_2$ define the orbital plane, the radius vector makes the angle u with $\hat{L}_1$, so that

$$\cos u = \hat{r} \cdot \hat{L}_1$$

$$\sin u = \hat{r} \cdot \hat{L}_2$$

PREVIOUS PAGE IS BLANK

[The formulation here differs from the customary one in terms of the basis defined by $\hat{N}$ (the vector along the line of nodes), $\hat{h}$, and $\hat{h} \times \hat{N}$. When the inclination is small, $\hat{N}$ and $\hat{h} \times \hat{N}$ are ill-defined, a disadvantage not suffered by $\hat{L}_1$ and $\hat{L}_2$. The change entails also using the longitudes (u, α, λ) instead of the anomalies (f, E, ℓ)].

From here on the algorithm is straightforward:

$$\left.\begin{aligned} h_x &= yv_z - zv_y \\ h_y &= zv_x - xv_z \\ h_z &= xv_y - yv_x \end{aligned}\right\}$$

$$h = \sqrt{h_x^2 + h_y^2 + h_z^2}$$

$$I_S = -h_x/h$$

$$I_C = -h_y/h$$

$$\rho = h_z/h$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\cos u = (x + zI_S/(1+\rho))/r$$

$$\sin u = (y + zI_C/(1+\rho))/r$$

$$e \cos f = (h^2/\mu r) - 1$$

$$e \sin f = (h/\mu r)(xv_x + yv_y + zv_z)$$

$$e^2 = (e \sin f)^2 + (e \cos f)^2$$

$$\zeta = \sqrt{1-e^2}$$

$$e \cos \tilde{\omega} = e \cos f \cos u + e \sin f \sin u$$

$$e \sin \tilde{\omega} = e \cos f \sin u - e \sin f \cos u$$

$$\frac{h^2}{\mu r} \cos \alpha = \cos u + e \cos \tilde{\omega} + e \sin \tilde{\omega} . e \sin f/(1 + \zeta)$$

$$\frac{h^2}{\mu r} \sin \alpha = \sin u + e \sin \tilde{\omega} - e \cos \tilde{\omega} . e \sin f/(1 + \zeta)$$

$$\alpha = \tan^{-1} \frac{(h^2/\mu r) \sin \alpha}{(h^2/\mu r) \cos \alpha}$$

$$L = h/\zeta$$

$$\lambda = \alpha - e \cos \tilde{\omega} \sin \alpha + e \sin \tilde{\omega} \cos \alpha$$

$$\xi = \sqrt{2L/(1+\zeta)}\ e \cos \tilde{\omega}$$

$$\eta = - \sqrt{2L/(1+\zeta)}\ e \sin \tilde{\omega}$$

$$\sigma = \sqrt{2L\zeta/(1+\rho)}\ I_C$$

$$\tau = \sqrt{2L\zeta/(1+\rho)}\ I_S$$

TRANSFORMATION FROM ELLIPTIC ELEMENTS TO OSCULATING CANONICAL VARIABLES

[This is the first section of the ENTER subroutine (lines 134-145) invoked when the KEY argument is 2.]

$$L = \sqrt{\mu a}$$

$$\lambda = \ell + g + h$$

$$\zeta = \sqrt{1 - e^2}$$

$$\xi = \sqrt{2L(1 - \zeta)} \cos (g + h)$$

$$\eta = - \sqrt{2L (1 - \zeta)} \sin (g + h)$$

$$\sigma = 2 \sqrt{L\zeta} \sin i/2 \cos h$$

$$\tau = - 2 \sqrt{L\zeta} \sin i/2 \sin h$$

TRANSFORMATION FROM OSCULATING CANONICAL VARIABLES TO ELLIPTIC ELEMENTS

[This is in subroutine OSTCOR (lines 194-203).] See the last section in this Appendix for the calculation of the intermediate quantities used below.

$$a = L^2/\mu$$

$$e = \sqrt{1 - \zeta^2}$$

$$i = \cos^{-1}\rho$$

$$h = \tan^{-1} (- \tau/\sigma)$$

$$g + h = \tan^{-1} (-\eta/\xi)$$

$$\ell = \lambda - g - h$$

TRANSFORMATION FROM OSCULATING CANONICAL VARIABLES TO CARTESIAN COORDINATES

[This is subroutine OSTCOR (lines 205-212).] Again the intermediate quantities described in the next section are used.

$$r = (1 - e \cos \tilde{\omega} \cos \alpha - e \sin \tilde{\omega} \sin \alpha)L^2/\mu$$

$$x = r((1 - \tau^2/2L\zeta) \cos u - (\sigma \tau/2L\zeta) \sin u)$$

$$y = r(-(\sigma \tau/2L\zeta) \cos u + (1 - \sigma^2/2L\zeta) \sin u)$$

$$z = r\ (\sigma \sin u + \tau \cos u)\sqrt{(1+\rho)/2L\zeta}$$

$$v_x = ((\alpha - \lambda)\ x + \zeta\ (-\ \sigma z \sqrt{(1+\rho)/2L\zeta} - \rho y))L/r^2$$

$$v_y = ((\alpha - \lambda)\ y + \zeta\ (\rho x + \tau z\ \sqrt{(1+\rho)2L\zeta}))L/r^2$$

$$v_z = ((\alpha - \lambda)\ z + \zeta\ (-\ \tau y + \sigma x)\sqrt{\ (1 + \rho)2L\zeta})L/r^2$$

CALCULATION OF INTERMEDIATE QUANTITIES FROM CANONICAL VARIABLES

(This is the subroutine INTERM.)

$$\zeta = 1 - (\xi^2 + \eta^2)/2L$$

$$\rho = 1 - (\sigma^2 + \tau^2)2L\zeta$$

$$e \cos \tilde{\omega} = \xi \sqrt{(1 + \zeta)/2L}$$

$$e \sin \tilde{\omega} = -\ \eta \sqrt{(1 + \zeta)/2L}$$

Solve for α, by Newton-Raphson, the Kepler equation:

$$\lambda = \alpha - e \cos \tilde{\omega} \sin \alpha + e \sin \tilde{\omega} \cos \alpha$$

$$u = \alpha + 2 \tan^{-1} ((\alpha - \lambda)/(1 + \zeta - e \cos \tilde{\omega} \cos \alpha - e \sin \tilde{\omega} \sin \alpha))$$

$$e \sin f = e \cos \tilde{\omega} \sin u - e \sin \tilde{\omega} \cos u$$

$$e \cos f = e \cos \tilde{\omega} \cos u + e \sin \tilde{\omega} \sin u$$

$$c_1 = \sqrt{(1 + \zeta)/2L}$$

$$c_2 = \sqrt{(1 + \rho)2L\zeta}$$

$$\gamma_2' = 12\ J_2\ (\mu\ R_e)^2/(2L\zeta)^4$$

APPENDIX D

DIFFERENTIAL RELATIONSHIPS AND THE TRANSFORMATION BETWEEN OSCULATING AND MEAN CANONICAL VARIABLES

DIFFERENTIAL RELATIONSHIPS AND THE TRANSFORMATION BETWEEN OSCULATING AND MEAN CANONICAL VARIABLES

The first-order determining function, whose partial derivatives give the transformation between mean and osculating variables, is given by

$$S_1 = 6\,(\mu R_e)^2\; J_2/(2L\zeta)^3 \Big\{ (-\;1 + 3\;\rho^2) U_\ell \;+$$

$$+\; (\;(1 + \rho)/4L\zeta)(\sigma^2 - \tau^2)\; S_{up} + 2\;\sigma\tau\; C_{up}) \Big\}$$

$$U_\ell = f - \ell + e \sin f$$

$$S_{up} = \sin 2u + e \sin (u + \tilde{\omega}) + (e/3) \sin (3u - \tilde{\omega})$$

$$C_{up} = \cos 2u + e \cos (u + \tilde{\omega}) + (e/3) \cos (3u - \tilde{\omega})$$

The advantage of this dissection is that none of the last three quantities defined depend upon σ or τ. Quantities such as e, f, and ℓ are to be considered as expressed in terms of the canonical variables being used. Hence, it is necessary to derive a miscellany of differential relationships that will be needed in the process.

Since $\zeta^2 = 1 - e^2$ and $1 - \zeta = (\xi^2 + \eta^2)/2L$,

$$de = -\;\zeta d\;\zeta/e = \frac{\zeta e}{1+\zeta}\left(\frac{2(\xi\; d\xi + \eta\; d\eta)}{\xi^2 + \eta^2} - \frac{dL}{L}\right)$$

Also, $d\tilde{\omega} = d(\tan^{-1}\;(-\;\eta/\xi)) = \dfrac{\eta d\xi - \xi d\eta}{\xi^2 + \eta^2}$

From the usual Keplerian orbit relationships:

$$\ell = E - e \sin E$$

$$r/a \sin f = \zeta \sin E$$

$$r/a \cos f = \cos E - e$$

PREVIOUS PAGE IS BLANK

Differentiation, substitution, and rearrangement yields

$$df = (Z^2/\zeta^3)\, d\ell + ((Z + 1) \sin f)/\zeta^2)de$$

where $Z = \zeta^2 \; a/r = 1 + e \cos f$.

t will also be convenient to use

$$e_c = \sqrt{(1 + \zeta)/2L}\; \xi = c_1 \; \xi$$

$$e_s = - c_1 \; \eta$$

rom which by differentiation:

$$de_c = c_1 \; d\xi + e_c/(2L(1 + \zeta))\; (- 2 \zeta \; dL - \xi \; d\xi - \eta \; d\eta)$$

$$de_s = - c_1 \; d\eta + e_s/(2L(1 + \zeta))\; (- 2 \zeta \; dL - \xi \; d\xi - \eta \; d\eta)$$

f the foregoing are applied,

$$dU_\ell = df(1 + e \cos f) - d\ell + \sin f \; de$$

$$= (Z^3/\zeta^3 - 1)\,(d\lambda - d\tilde{\omega}) + (Z^2 + Z + \zeta^2) \sin f \; de/\zeta^2$$

$$= \frac{1}{\zeta^2 + \eta^2}\; [(Z^3/\zeta^3 - 1)(- \eta \; d\xi + \xi \; d\eta) + 2\, \frac{(Z^2 + Z + \zeta^2)}{\zeta(1 + \zeta)} e \sin f(\xi \, d\xi + \eta \, d\eta)]$$

$$+ (Z^3/\zeta^3 - 1)d\lambda - \frac{2(Z^2 + Z + \zeta^2)}{L\zeta \; (1 + \zeta)}\; e \sin f \; dL$$

he coefficients of $d\xi$ and $d\eta$ are not convenient for computing, since at low eccentricities ey become the ratios of small quantities. It is expedient to transform them using the easily erived relations:

$$- \eta \cos f + \xi \sin f = \sqrt{2L/(1 + \zeta)}\; e \sin u$$

$$\eta \sin f + \xi \cos f = \sqrt{2L/(1 + \zeta)}\; e \cos u$$

arriving finally at

$$dU_\ell = \frac{1 + Z + Z^2}{2L\zeta^3} [2L(1 + \zeta) \sin u \, d\xi + \cos u \, d\eta)$$

$$- \frac{2\zeta+1}{1+\zeta} e \sin f \, (\xi d\xi + \eta d\eta)] + \frac{1 + \zeta + \zeta^2}{2L\zeta^3} (- \eta \, d\xi + \xi \, d\eta)$$

$$- (\xi \, d\xi + \eta \, d\eta)/L\zeta - \frac{Z^2 + Z + \zeta^2}{L\zeta \, (1 + \zeta)} e \sin f \, dL + (Z^3/\zeta^3 - 1)d\lambda$$

Next C_{up} may be written = $\cos 2u + e_c(\cos u + (1/3) \cos 3u)$

$$+ e_s \, (- \sin u + (1/3) \sin 3u)$$

so that $dC_{up} = - (2 \sin 2u + e_c \, (\sin u + \sin 3u) + e_s \, (\cos u - \cos 3u) \, (df + d\tilde{\omega})$

$$+ (\cos u + (1/3) \cos 3 u) \, de_c + (-\sin u + (1/3) \sin 3u)de_s$$

Note that $2 \sin 2 u + e_c \, (\sin u + \sin 3u) + e_s \, (\cos u - \cos 3u) = 2Z \sin 2 u$;

substitute, rearrange, and make the same transformation as for U_ℓ. The result is

$$dC_{up} = - (Z \sin 2u/L\zeta^3) \, [(Z + 1) \frac{1 + \zeta}{c_1} (\sin u \, d\xi + \cos u \, d\eta)$$

$$- \frac{1 + 2\zeta}{1 + \zeta} \quad e \sin f(\xi \, d\xi + \eta \, d\eta)) + (1 + \zeta + \zeta^2) \, (- \eta \, d\xi + \xi \, d\eta)]$$

$$+ c_1 \, [\cos u + (1/3) \cos 3 u)d\xi + (\sin u - (1/3) \sin 3 u)d\eta]$$

$$- \frac{C_{up} - \cos 2 u}{2L(1 + \zeta)} (\xi \, d\xi + \eta \, d\eta) + \frac{dL}{L\zeta \, (1 + \zeta)} [2Z \, (Z + 1) \sin 2u . e \sin f$$

$$- \zeta^2 \, (C_{up} - \cos 2 u)] - 2Z^3 \sin 2 u \, d\lambda/\zeta^3$$

DISTRIBUTION

	Copies
Library of Congress ATTN: Gift and Exchange Division Washington, DC 20540	4
Defense Mapping Agency U. S. Navy Observatory Washington, DC 20305	1
U. S. Naval Observatory Nautical Almanac Office Washington, DC 20305	2
Naval Space Surveillance System Dahlgren, VA 22448	1
The University of Texas Astronomy Department Austin, TX	2
Internal Distribution:	
E31 (GIDEP)	1
E411 (Hall)	1
E431	10
K10	5
K104	50

$$\frac{\partial \beta}{\partial L} = c_1 \; (x'\tau + y'\sigma) \; 2v_2/(2L\zeta) + c_1 c_2 z' \; (- \; 2v_2 \; \rho)/(1 + \rho) - 2 \; \beta\zeta/(2L(1 + \zeta))$$

$$\frac{\partial \gamma}{\partial \xi} = (2 \; \xi/(2L\zeta))(- \; c_2 \; \rho \; (x'\tau + y'\sigma) \; /(1 + \rho) - \; z' \; (1 - \rho))$$

$$\frac{\partial \gamma}{\partial \eta} = (2\eta/(2L\zeta))(- \; c_2 \; \rho(x'\tau + y'\sigma)/(1 + \rho) - z' (1 - \rho))$$

$$\frac{\partial \gamma}{\partial \sigma} = c_2 \sigma \; (x'\tau + y'\sigma)/(2L\zeta \; (1 + \rho)) - c_2 y' - 2z'\sigma/(2L\zeta)$$

$$\frac{\partial \gamma}{\partial \tau} = c_2 \tau \; (x'\tau + y'\sigma)/(2L\zeta \; (1 + \rho)) - c_2 x' - 2z'\tau/(2L\zeta)$$

$$\frac{\partial \gamma}{\partial L} = 2c_2 \rho \; (x'\tau + y'\sigma)/(2L\zeta \; (1 + \rho)) + 2z' \; (1 - \rho)/(2L\zeta)$$

$$\frac{\partial \zeta}{\partial \xi} = - \; \xi/L$$

$$\frac{\partial \zeta}{\partial \eta} = - \; \eta/L$$

$$\frac{\partial \zeta}{\partial L} = (1 - \zeta)/L$$

$$\frac{\partial a}{\partial L} = 2L/\mu$$

The other partial derivatives in the matrix are zero.

$$\zeta = 1 - (\xi^2 + \eta^2)/2L, \qquad a = L^2/\mu$$

we may write

$$\alpha_x = c_1\ (\xi - v_1\ \tau) \qquad \alpha_y = c_1\ (-\ \eta - v_1\ \sigma) \qquad \alpha_z = c_2\ v_1\ (2L\zeta)$$

$$\beta_x = c_1\ (-\ \eta - v_2\ \tau) \qquad \beta_y = c_1\ (-\ \xi - v_2\ \sigma) \qquad \beta_z = c_2\ v_2\ (2L\zeta)$$

$$\gamma_x = -\ c_2\ \tau \qquad \gamma_y = -\ c_2\ \sigma \qquad \gamma_x = \rho$$

and easily but tediously differentiate these to find

$$\frac{\partial\alpha}{\partial\xi} = c_1\ x' - c_1\ (\tau + 2v_1\ \xi)(x'\tau + y'\sigma)/(2L\zeta) + c_1 c_2 z'\ (\tau + 2v_1\ \xi\ \rho/(1+\rho)) - \alpha\xi/(2L(1+\zeta))$$

$$\frac{\partial\alpha}{\partial\eta} = -\ c_1 y' + c_1\ (\sigma - 2v_1\ \eta)(x'\tau + y'\sigma)/(2L\zeta) + c_1 c_2 z'\ (-\ \sigma + 2v_1\ \eta\ \rho/(1+\rho)) - \alpha\eta/(2L(1+\zeta))$$

$$\frac{\partial\alpha}{\partial\sigma} = c_1\ (x'\tau + y'\sigma)\ \eta/(2L\zeta) - c_1\ y'v_1 + c_1 c_2 z'\ (-\ \eta - \sigma v_1/(1+\rho))$$

$$\frac{\partial\alpha}{\partial\tau} = -\ c_1\ (x'\tau + y'\sigma)\ \xi/(2L\zeta) - c_1 x'v_1 + c_1 c_2 z'\ (\xi - \tau\ v_1/(1+\rho))$$

$$\frac{\partial\alpha}{\partial L} = c_1\ (x'\tau + y'\sigma)2v_1/(2L\zeta) + c_1\ c_2\ z'\ (-\ 2v_1\ \rho)/(1+\rho) - 2\ \sigma\zeta/(2L(1+\zeta))$$

$$\frac{\partial\beta}{\partial\xi} = -\ c_1 y' + c_1\ (\sigma - 2v_2\xi)(x'\tau + y'\sigma)/(2L\zeta) + c_1 c_2 z'(-\ \sigma + 2v_2\ \xi\rho/(1+\rho)) - \beta\xi/(2L(1+\xi))$$

$$\frac{\partial\beta}{\partial\eta} = -\ c_1 x' + c_1\ (\tau - 2v_2\eta)(x'\tau + y'\sigma)/(2L\zeta) + c_1 c_2 z'\ (-\ \tau + 2v_2\ \eta\ \rho/(1+\rho)) - \beta\eta/(2L(1+\zeta))$$

$$\frac{\partial\beta}{\partial\sigma} = c_1\ (x'\tau + y'\sigma)\ \xi/(2L\zeta) - c_1\ y'v_2 + c_1 c_2 z'\ (-\ \xi - v_2\ \sigma)/(1+\rho))$$

$$\frac{\partial\beta}{\partial\tau} = c_1\ (x'\tau + y'\sigma)\ \eta\ /(2L\zeta) - c_1 x'v_2 + c_1 c_2 z'\ (-\eta - v_2\ \tau/(1+\rho))$$

CALCULATION OF THE JACOBIAN MATRIX

To complete the differentiation of the Hamiltonian with respect to the mean variables, we need the Jacobian matrix

$$\frac{J(\alpha, \beta, \gamma, \zeta, a)}{J(\eta, \xi, \tau, \sigma, L)}$$

for the three cases: zonal harmonics, lunar attraction, and solar attraction. It is clear from THE HAMILTONIAN IN TERMS OF MEAN VARIABLES section that we can subsume the three into one by writing

$$\alpha = \alpha_x x' + \alpha_y y' + \alpha_z z'$$

$$\beta = \beta_x x' + \beta_y y' + \beta_z z'$$

$$\gamma = \gamma_x x' + \gamma_y y' + \gamma_z z'$$

where the unit vector

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \text{ is } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ for the zonals, } \begin{pmatrix} x'_M \\ y'_M \\ z'_M \end{pmatrix} \text{ for the moon, and } \begin{pmatrix} x'_S \\ y'_S \\ z'_S \end{pmatrix} \text{ for the sun}$$

(and constant in all three cases), and ζ, a, α_x, α_y, α_z, β_x, β_y, β_z, γ_x, γ_y, and γ_z are the same for all three cases.

If we define

$$v_1 = (\xi\tau - \eta\sigma)/2L\zeta$$

$$v_2 = -(\xi\sigma + \eta\tau)/2L\zeta$$

and remember that $c_1 = \sqrt{(1+\zeta)/2L}$, $\qquad c_2 = \sqrt{(1+\rho)/2L\zeta}$

PREVIOUS PAGE IS BLANK

APPENDIX G

CALCULATION OF THE JACOBIAN MATRIX

Thus,

$$C_m(\alpha, \beta) = \alpha C_{m-1}(\alpha, \beta) - \beta S_{m-1}(\alpha, \beta)$$

$$S_m(\alpha, \beta) = \alpha S_{m-1}(\alpha, \beta) + \beta C_{m-1}(\alpha, \beta)$$

and the starting values are

$$C_0(\alpha, \beta) = 1 \qquad S_0(\alpha, \beta) = 0$$

Differentiation from the definition is

$$\frac{\partial C_m}{\partial \alpha}(\alpha, \beta) = m\, C_{m-1}(\alpha, \beta)$$

$$\frac{\partial C_m}{\partial \beta}(\alpha, \beta) = -\, m\, S_{m-1}(\alpha, \beta)$$

Derivation of the results below is parallel to that in the section above.

$$A^m_{n+1}(x) = (-)^m \frac{(n-m)!}{n!} \frac{P^m_n(x)}{x^n e^m}$$

Upon substitution for P^m_n (see the previous section),

$$A^m_{n+1}(x) = (-)^m \frac{(n-m)!}{n!} x^{-n+m} Q^m_n(x)$$

The recursion relation is

$$n(n+1) A^m_{n+2}(x) = n(2n+1) A^m_{n+1}(x) - \frac{(n+m)(n-m)}{x^2} A^m_n(x)$$

with starting values

$$A^m_m(x) = 0 \qquad\qquad A^m_{m+1}(x) = (-)^m \frac{Q^m_m(x)}{m!}$$

and starting recursion

$$A^m_{m+1}(x) = -\frac{2m-1}{m} A^{m-1}_m(x); \qquad\qquad A^0_1(x) = 1$$

Differentiation, from the function definition, is

$$\frac{d}{dx} A^m_{n+1}(x) = -\frac{n-m}{x}\left(A^m_{n+1}(x) + A^{m+1}_{n+1}(x)\right)$$

RECURSION ON $C_m(\alpha, \beta)$ AND $S_m(\alpha, \beta)$

These coefficients are defined by

$$(\alpha + i\beta)^m = C_m(\alpha, \beta) + i\, S_m(\alpha, \beta)$$

so that

$$C_m(\alpha, \beta) + i\, S_m(\alpha, \beta) = (\alpha + i\,\beta)(C_{m-1}(\alpha, \beta) + S_{m-1}(\alpha, \beta))$$

RECURSION ON B_n^m (x)

These are almost the coefficients defined by Cefola and Broucke (Reference F-1, Equations 68-80), but multiplied by x^{2n+1}, which simplifies the calculations required.

$$B_{n+1}^m(x) = \frac{n!}{(n+m)!}\,\frac{x^{n+1}\,P_n^m(x)}{e^m} \qquad \text{(eq. 76)}$$

(Note that this P_n^m has a difference: it is as defined in Reference F-2:

$$P_n^m(x) = (x^2 - 1)^{m/2}\,Q_n^m(x), \text{ not } (1 - x^2)^{m/2}.$$

$$B_{n+1}^m(x) = \frac{n!}{(n+m)!}\,x^{n+m+1}\,Q_n^m(x)$$

Now the recursion follows from that for Q_n^m above.

$$(n-m+1)(n+m+1)\,B_{n+2}^m(x) = (n+1)\,x^2\,[(2n+1)\,B_{n+1}^m(x) - nB_n^m(x)]$$

with starting values $B_m^m(x) = 0$, $B_{m+1}^m(x) = x^{2m+1}/2^m$,

and the starting recursion $B_{m+1}^m(x) = (x^2/2)\,B_m^{m-1}(x)$, $B_1^0(x) = x$

Differentiation of the function, from its definition, is

$$\frac{d}{dx}B_{n+1}^m(x) = \frac{n+m+1}{x}\,(B_{n+1}^m(x) + B_{n+1}^{m+1}(x))$$

RECURSION ON A_n^m (x)

These are the coefficients described by Cefola and Broucke (Reference F-1, Equations 54-67), but with an additional factor of $(-1)^m$.

[F-1] Paul J. Cefola and Roger Broucke, AIAA Paper No. 75-9 (no date)

[F-2] Milton Abramowitz and Irene A. Stegun, Eds., "Handbook of Mathematical Functions" (Dover).

RECURSION ON $Q_n^m(\gamma)$

The function is defined by $Q_n^m(\gamma) = \frac{d^m}{d\gamma^m} P_n(\gamma)$ with P_n the Legendre polynomial of the first kind. The recursion relationship is easily obtained by differentiating the recursion relationship for the Legendre polynomial:

$$(n+2-m)\,Q_{n+2}^{m}(\gamma) = (2n+3)\,\gamma\,Q_{n+1}^{m}(\gamma) - (n+m+1)Q_n^m(\gamma)$$

with starting values $Q_m^{m+1}(\gamma) = 0$

$$Q_m^m(\gamma) = (2m-1)!!$$

$$Q_{m+1}^{m}(\gamma) = \gamma\,(2m+1)!!$$

where $(2m+1)!! = 1.3.5.7\ldots.(2m+1)$. The starting values may themselves be determined recursively:

$$Q_m^m(\gamma) = (2m-1)\,Q_{m-1}^{m-1}(\gamma)$$

$$Q_{m+1}^{m}(\gamma) = (2m+1)\gamma\,Q_m^m(\gamma)$$

$$Q_0^0(\gamma) = 1$$

Differentiation, from the function definition, is

$$\frac{d}{d\gamma} Q_n^m(\gamma) = Q_n^{m+1}(\gamma)$$

RECURSION OF D_n

D_n^m is defined by $\frac{(n-m)!}{(n+m)!} Q_n^m(0)$ and its recursive relationships

follow from those of Q_n^m: $D_n^{m+2} = -\frac{n-m+1}{n+m+2} D_n^m$

Starting values are $D_m^m = \frac{1}{2^m\ m!}$, $D_{m+1}^{m} = 0$

and the former allows the recursion

$$D_m^m = \frac{1}{2m} D_{m-1}^{m-1}, \text{ with } D_0^0 = 1$$

PREVIOUS PAGE IS BLANK

APPENDIX F

RECURSIVE FUNCTIONS AND THEIR DERIVATIVES

THE SECOND-ORDER SHORT-PERIOD CORRECTION TO L

The last equation in the MEAN ELEMENTS: THE VON ZEIPEL TRANSFORMATION shows the transformation from L to L' with the second-order correction. The portions of this involving S_2 are calculated in the subroutine ELTRAN, lines 273-278. The term remaining, $(L^3/\mu^2)(F_2 - F_2^*)$, is more conveniently calculated in the subroutine DERIV, lines 621-643, where F_2^* is already available. The calculation of F_2, the instantaneous second-order portion of the Hamiltonian, is as follows:

(1) The zonal harmonics contribution to F_2.

$$(\Delta F)_{2Z} = - \sum_{n=3} (\mu/r)\ J_n\ (R_e/r)^n\ P_n\ (\sin \bar{\beta})$$

where $r = L^2\ \zeta^2/\mu Z$ and

$$\sin \bar{\beta} = \sin i \sin (f + g) = c_2\ (\sigma \sin u + \tau \cos u)$$

(2) The lunar contribution to F_2.

$$(\Delta F)_{2M} = \sum_{n=2} (\mu/r_M)\ ({}^r/r_M)^n\ P_n\ (\sin \bar{\beta}')$$

where r is as above and

$$\sin \bar{\beta}' = x'_M\ (\cos u - c_2\ \tau \sin \bar{\beta}/(1 + \rho))$$

$$+ y'_M\ (\sin u - c_2\ \sigma \sin \bar{\beta}/(1 + \rho) + z'_M \sin \bar{\beta}$$

The distance to the moon is r_M, and the components of the unit vector to the moon x'_M, y'_M, z'_M, in the coordinate frame $\hat{L}_1\ \hat{L}_2\ \hat{h}$, and $c_2 = \sqrt{(1 + \rho/L\zeta}$.

(3) The solar contribution to F_2 is exactly the same, except that the solar coordinates are substituted for lunar coordinates.

Since F_2 is second-order, it makes no difference whether it is calculated in terms of mean or osculating variables.

For a derivation of the formulas above see THE HAMILTONIAN IN TERMS OF MEAN VARIABLES section.

PREVIOUS PAGE IS BLANK

APPENDIX E

THE SECOND-ORDER SHORT-PERIOD CORRECTION TO L

$$dS_1 = \gamma_2' \left\{ L\zeta \left(-\frac{1}{3} + \rho^2 \right) dU_\ell + ((1 + \rho)/4)\ (\sigma^2 - \tau^2) dS_{up} + 2\ \sigma\tau\ dC_{up} \right)$$

$$+ [(1 + 2\rho - 5\rho^2)\ U_\ell +$$

$$((3 + 5\rho)/4L\ \zeta)\ ((\sigma^2 - \tau^2)\ S_{up} + 2\ \sigma\tau\ C_{up})]\ (dL - \xi\ d\xi - \eta\ d\eta)$$

$$- (2\ \rho\ U_\ell + (1/4L\zeta)\ ((\sigma^2 - \tau^2)\ S_{up} + 2\sigma\tau\ C_{up}))\ (\sigma\ d\sigma + \tau\ d\tau)$$

$$+ ((1 + \rho)/2)\ ((\sigma\ d\sigma - \tau\ d\tau)\ S_{up} + (\sigma\ d\tau + \tau d\sigma)\ C_{up}) \Big\}$$

The transformation between mean and osculating coordinates is taken from these equations and is contained in the subroutine ELTRAN, lines 215-271.

In exactly similar fashion,

$$dS_{up} = (Z \cos 2u/L\zeta^3)\ [(Z + 1)\ \frac{(1 + \zeta)}{c_1}\ (\sin u\ d\xi + \cos u\ d\eta)$$

$$- \frac{1 + 2\zeta}{1 + \zeta}\ e \sin f(\xi\ d\xi + \eta\ d\eta)) + (1 + \zeta + \zeta^2)\ (-\eta\ d\xi + \xi\ d\eta)\ ]$$

$$+ c_1\ [\sin u + (1/3) \sin 3\ u)d\xi - (\cos u - (1/3) \cos 3\ u)d\eta]$$

$$- \frac{S_{up} - \sin 2\ u}{2L(1 + \zeta)}\ (\xi\ d\xi + \eta\ d\eta) + \frac{dL}{L\zeta(1 + \zeta)}\ [-\ 2Z(Z + 1) \cos 2\ u,e \sin f$$

$$- \zeta^2\ (S_{up} - \sin 2\ u)] + 2Z^3 \cos 2\ u\ d\lambda/\zeta^3$$

There is more differentiation to be done. We may write

$$S_1 = \gamma_2'\ L\zeta\ [(-\ \frac{1}{3} + \rho^2)\ U_\zeta + \frac{1 + \rho}{4L}\ (\sigma^2 - \tau^2)\ S_{up} + 2\ \sigma\tau\ C_{up})]$$

$$= \gamma_2'\ (L\zeta)^4 \left[\left(-\ \frac{1}{3}\ \frac{1}{(L\zeta)^3} + \frac{(\rho L\zeta)^2}{(L\zeta)^5}\right) U_\zeta + \frac{1}{4}\left(\frac{1}{(L\zeta)^4} + \frac{\rho L\zeta}{(L\zeta)^5}\right) * \right.$$

$$\left. ((\sigma^2 - \tau^2)\ S_{up} + 2\ \sigma\tau\ C_{up}) \right]$$

and observe that $\gamma_2'\ (L\ \zeta)^4$ is a constant and that

$$L\zeta = L - (\xi^2 + \eta^2)/2$$

$$\rho L\zeta = L\zeta - (\sigma^2 + \tau^2)/2$$

so that differentiating S_1 with respect to $L\zeta$ and $\rho L\zeta$, and $L\zeta$ and $\rho L\zeta$ with respect to the variables yields the required derivatives.

END

FILMED

10-85

DTIC